

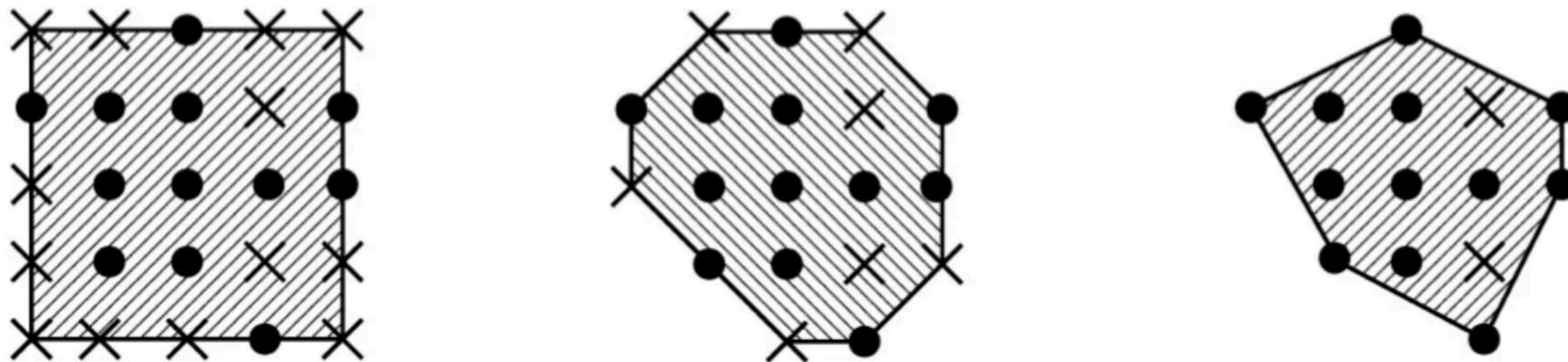
COSE312: Compilers

Lecture 15 – Semantic Analysis (3)

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2026 Spring

Relational Abstract Domains

- Intervals vs. Octagons vs. Polyhedra



- Focus: Core idea of the Octagon domain*

```
int a[10];  
x = 0; y = 0;
```

```
while (x < 9) {  
    x++; y++;  
}
```

```
a[y] = 0;
```

Octagon analysis

$x : [9,9]$

$y : [9,9]$

$x - y : [0,0]$

$x + y : [18,18]$

Interval analysis

$x : [9,9]$

$y : [0,\infty]$

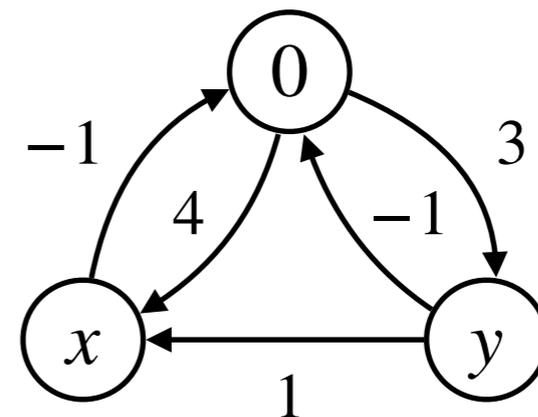
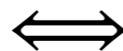
Difference Bound Matrix (DBM)

- A DBM represents a set of program states (N-dim points)

$$\gamma \left(\begin{bmatrix} 0 & 10 & +\infty \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \right) = \{(x, y) \mid 1 \leq x \leq 10, 0 \leq y, y - x \leq -1, x - y \leq 1\}$$

- A DBM can also be represented by a directed graph

$$\begin{array}{c} 0 \\ x \\ y \end{array} \begin{array}{ccc} & 0 & x & y \\ \left[\begin{array}{ccc} +\infty & 4 & 3 \\ -1 & +\infty & +\infty \\ -1 & 1 & +\infty \end{array} \right] \end{array}$$



Difference Bound Matrix (DBM)

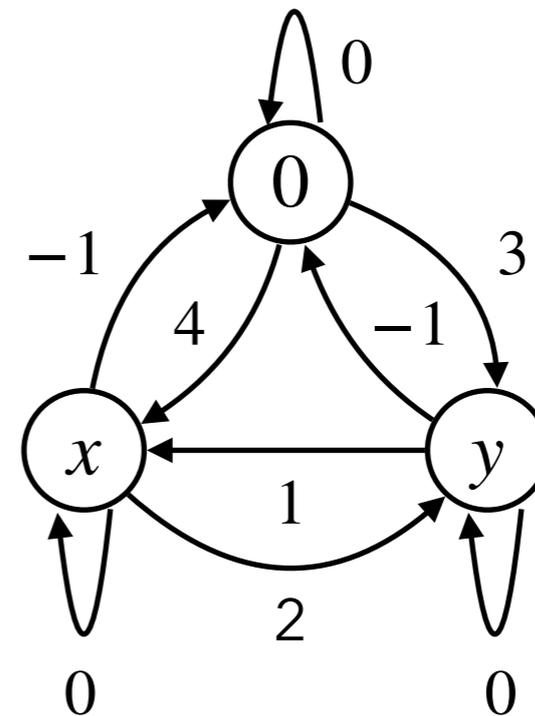
- Two different DBMs can represent the same set of points

$$\gamma \left(\begin{bmatrix} +\infty & 4 & 3 \\ -1 & +\infty & +\infty \\ -1 & 1 & +\infty \end{bmatrix} \right) = \gamma \left(\begin{bmatrix} 0 & 5 & 3 \\ -1 & +\infty & +\infty \\ -1 & 1 & +\infty \end{bmatrix} \right)$$

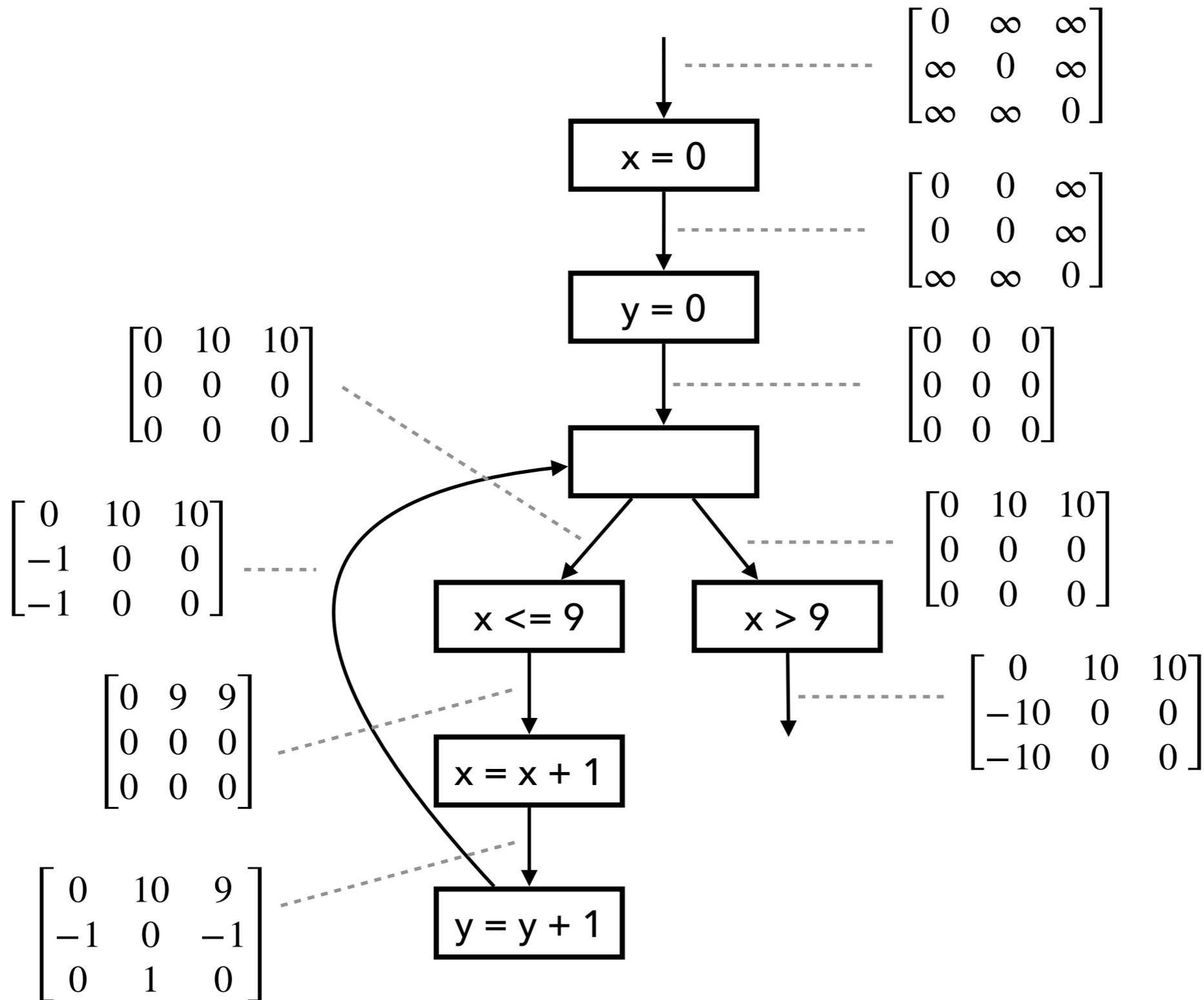
- Closure (normalization) via the Floyd-Warshall algorithm

$$\begin{bmatrix} +\infty & 4 & 3 \\ -1 & +\infty & +\infty \\ -1 & 1 & +\infty \end{bmatrix}^* = \begin{bmatrix} 0 & 4 & 3 \\ -1 & 0 & 2 \\ -1 & 1 & 0 \end{bmatrix}$$

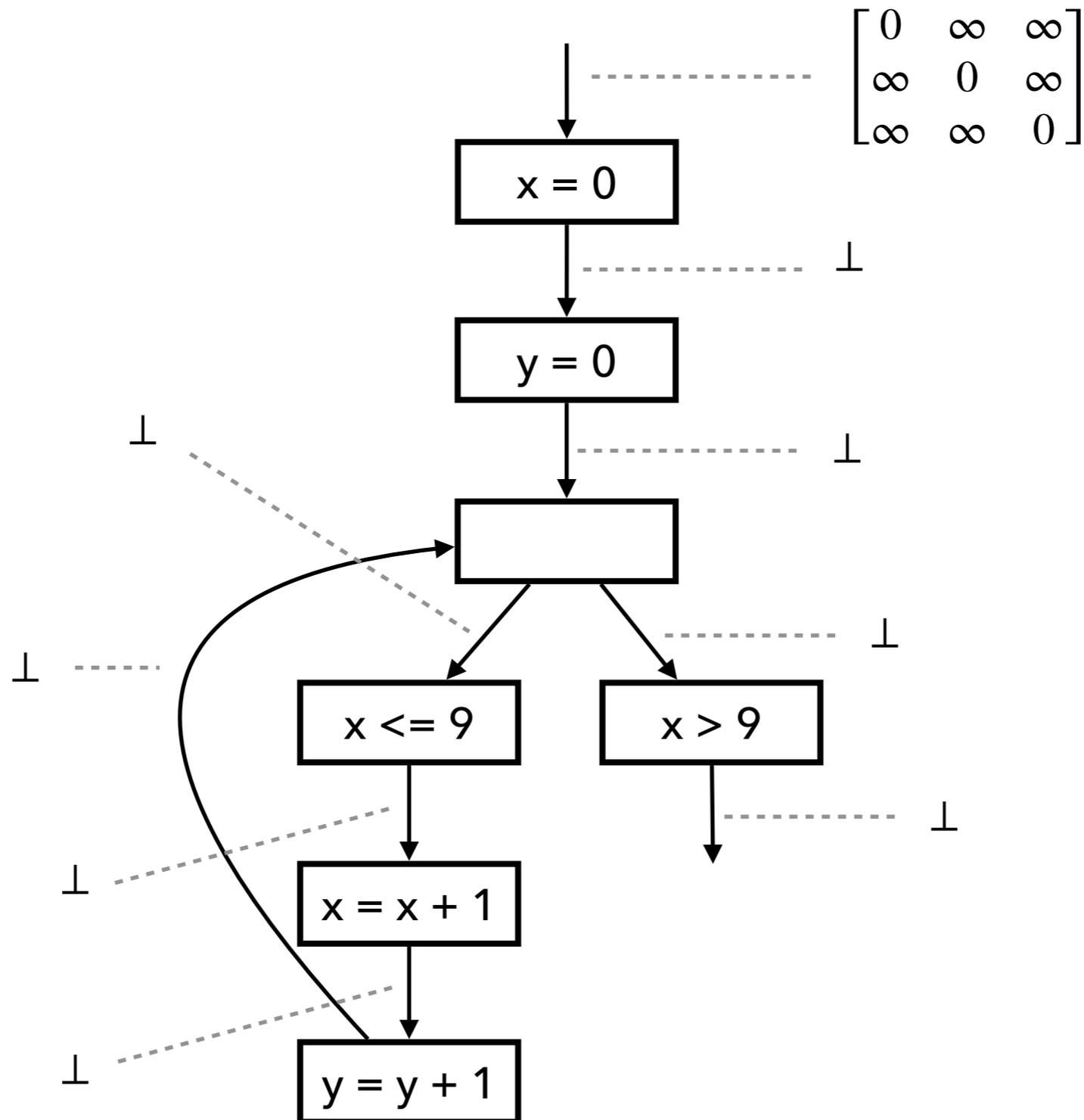
$$\begin{bmatrix} 0 & 5 & 3 \\ -1 & +\infty & +\infty \\ -1 & 1 & +\infty \end{bmatrix}^* = \begin{bmatrix} 0 & 4 & 3 \\ -1 & 0 & 2 \\ -1 & 1 & 0 \end{bmatrix}$$



Example



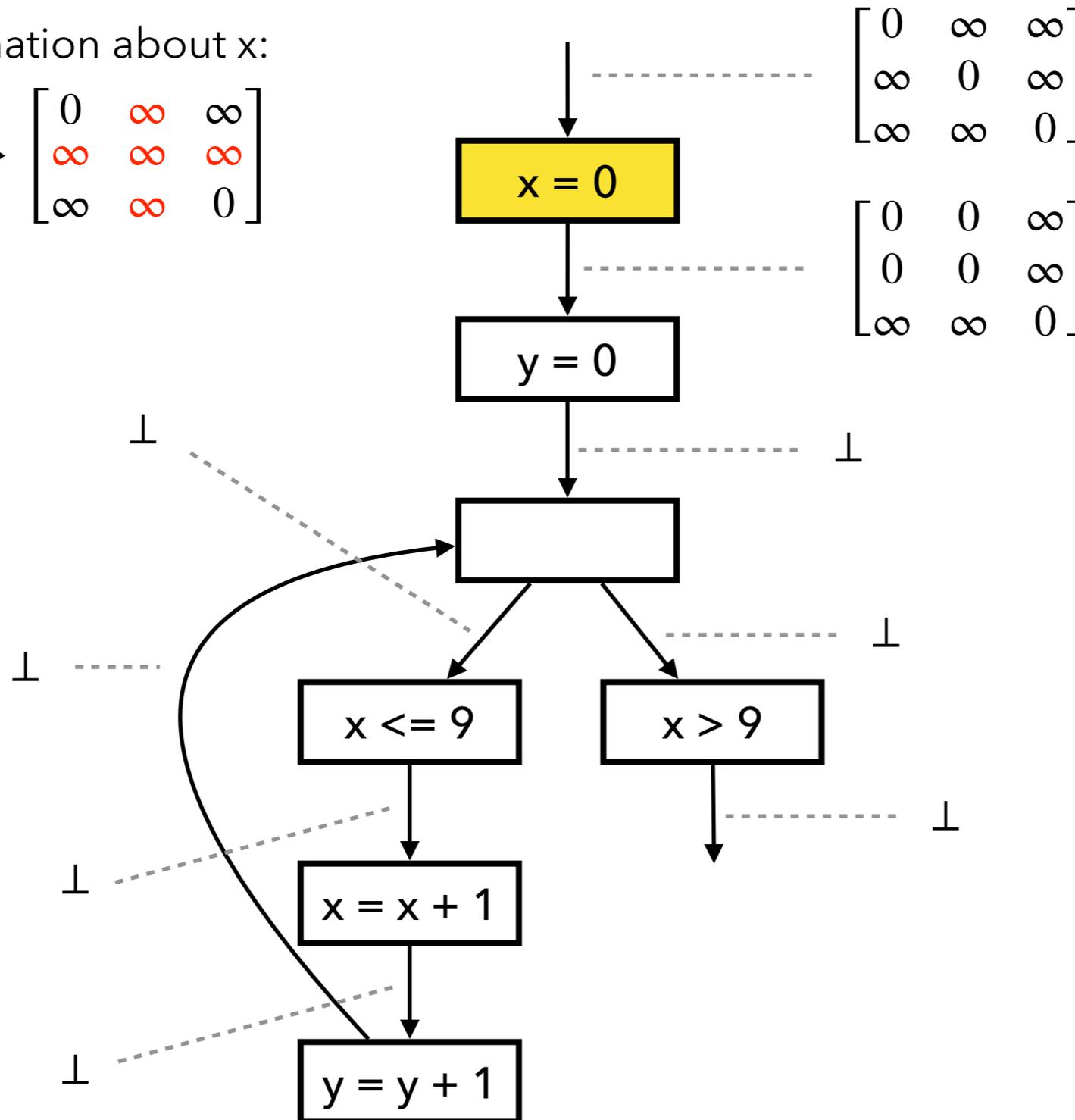
Fixed Point Comp. with Widening



Fixed Point Comp. with Widening

1. Remove information about x:

$$\begin{bmatrix} 0 & \infty & \infty \\ \infty & 0 & \infty \\ \infty & \infty & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & \infty & \infty \\ \infty & \infty & \infty \\ \infty & \infty & 0 \end{bmatrix}$$

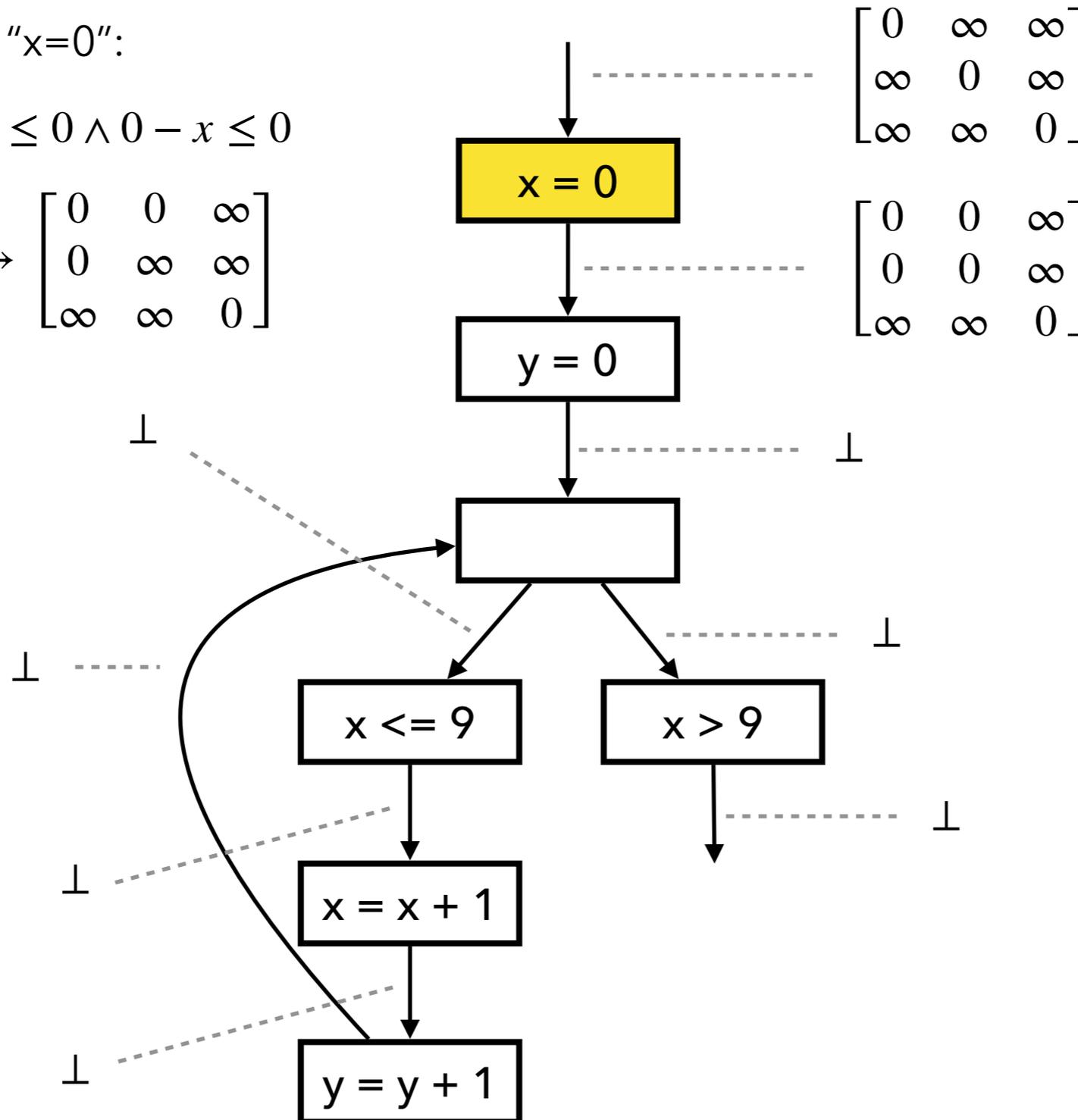


Fixed Point Comp. with Widening

2. Add constraint "x=0":

$$x = 0 \iff x - 0 \leq 0 \wedge 0 - x \leq 0$$

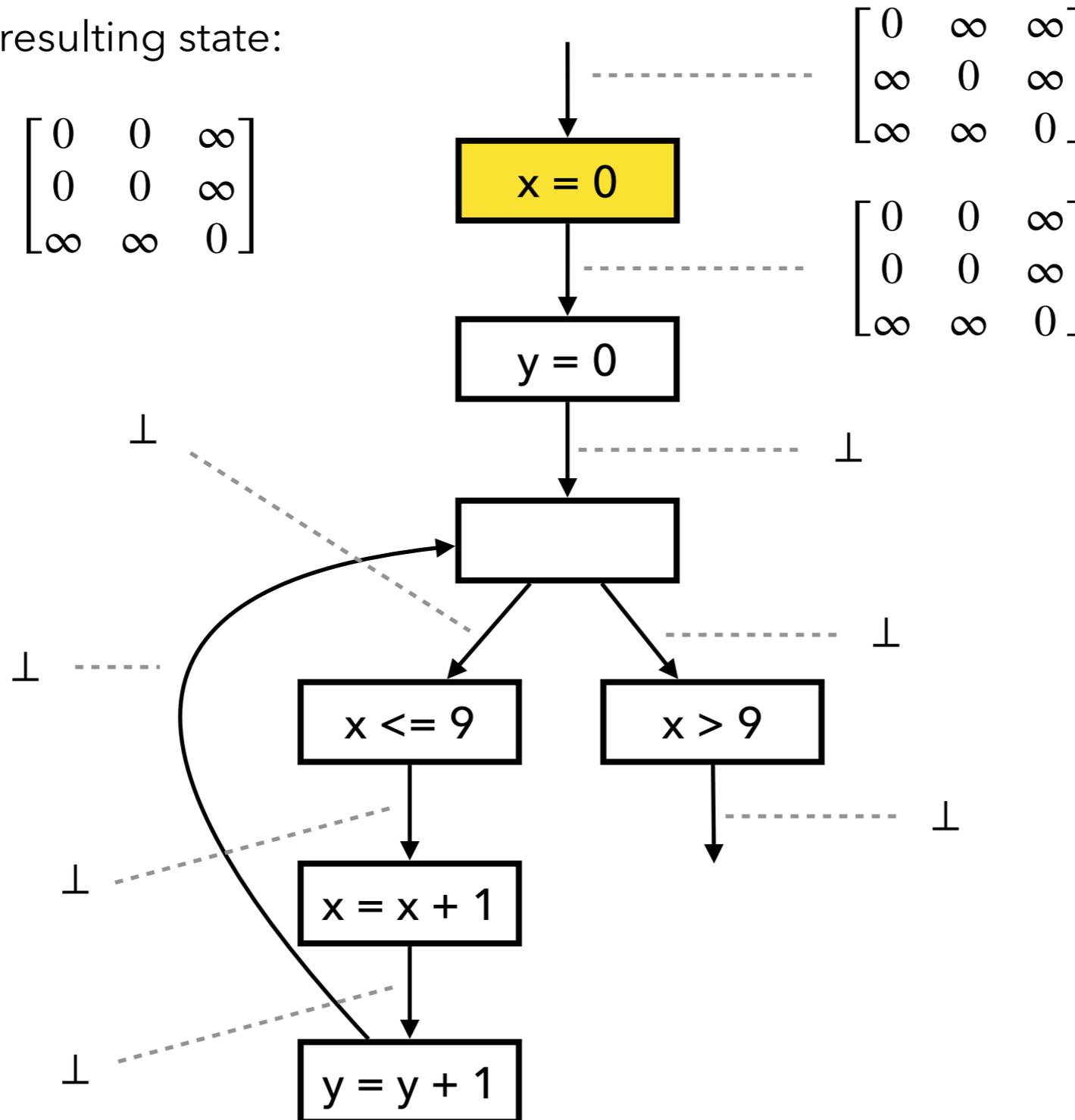
$$\begin{bmatrix} 0 & \infty & \infty \\ \infty & \infty & \infty \\ \infty & \infty & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & \infty \\ 0 & \infty & \infty \\ \infty & \infty & 0 \end{bmatrix}$$



Fixed Point Comp. with Widening

3. Normalize the resulting state:

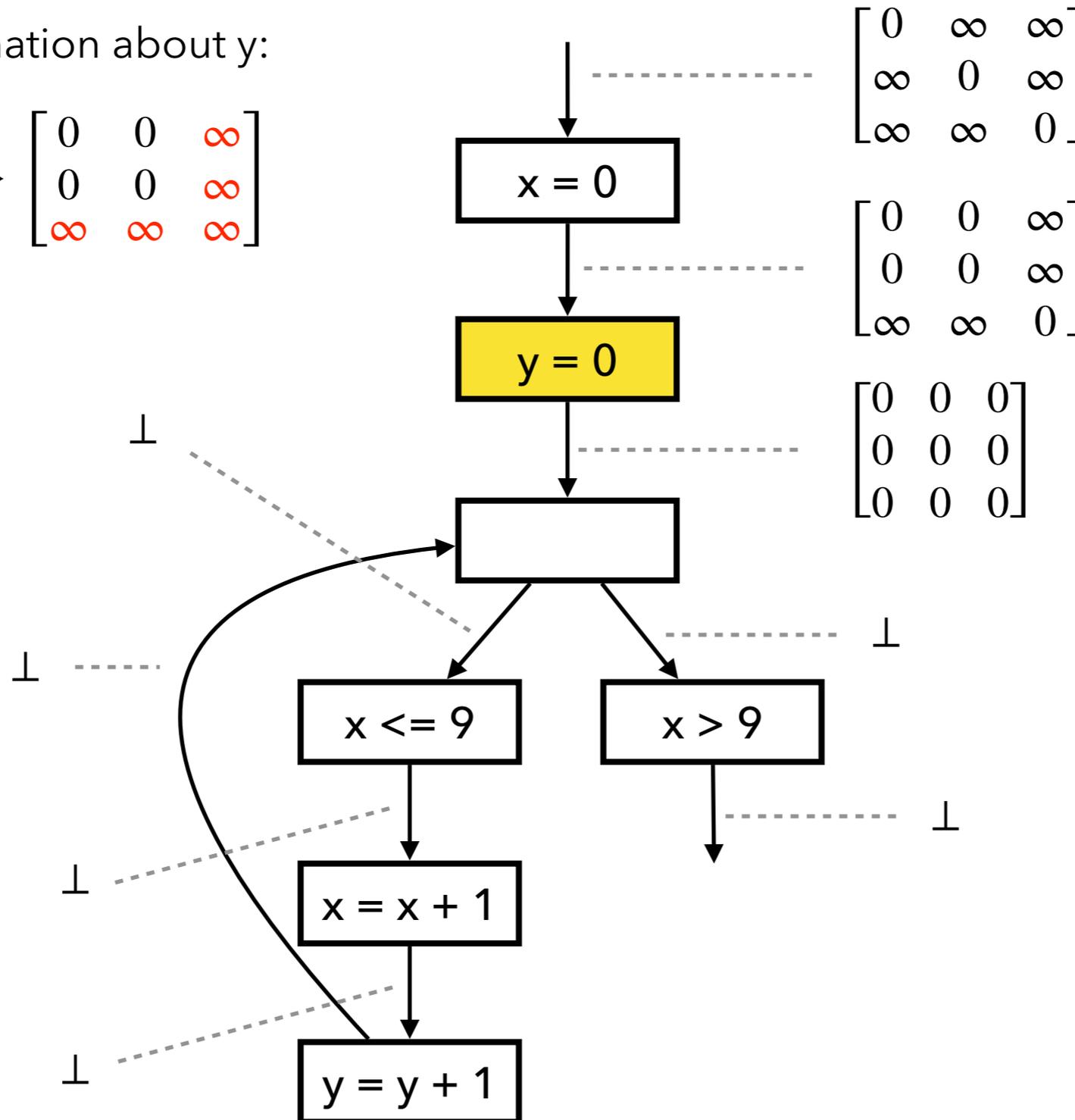
$$\begin{bmatrix} 0 & 0 & \infty \\ 0 & \infty & \infty \\ \infty & \infty & 0 \end{bmatrix}^* = \begin{bmatrix} 0 & 0 & \infty \\ 0 & 0 & \infty \\ \infty & \infty & 0 \end{bmatrix}$$



Fixed Point Comp. with Widening

1. Remove information about y:

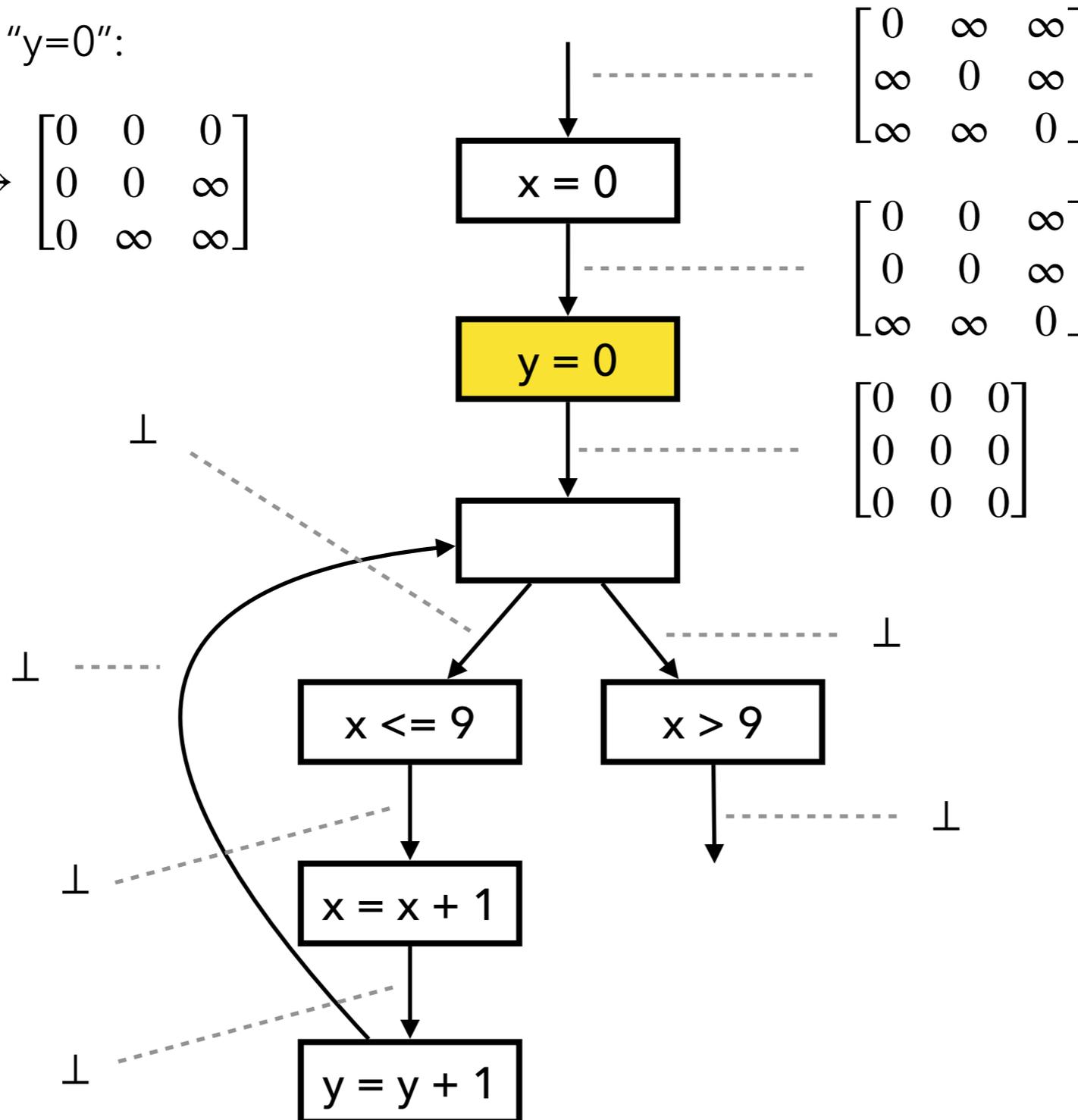
$$\begin{bmatrix} 0 & 0 & \infty \\ 0 & 0 & \infty \\ \infty & \infty & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & \infty \\ 0 & 0 & \infty \\ \infty & \infty & \infty \end{bmatrix}$$



Fixed Point Comp. with Widening

2. Add constraint "y=0":

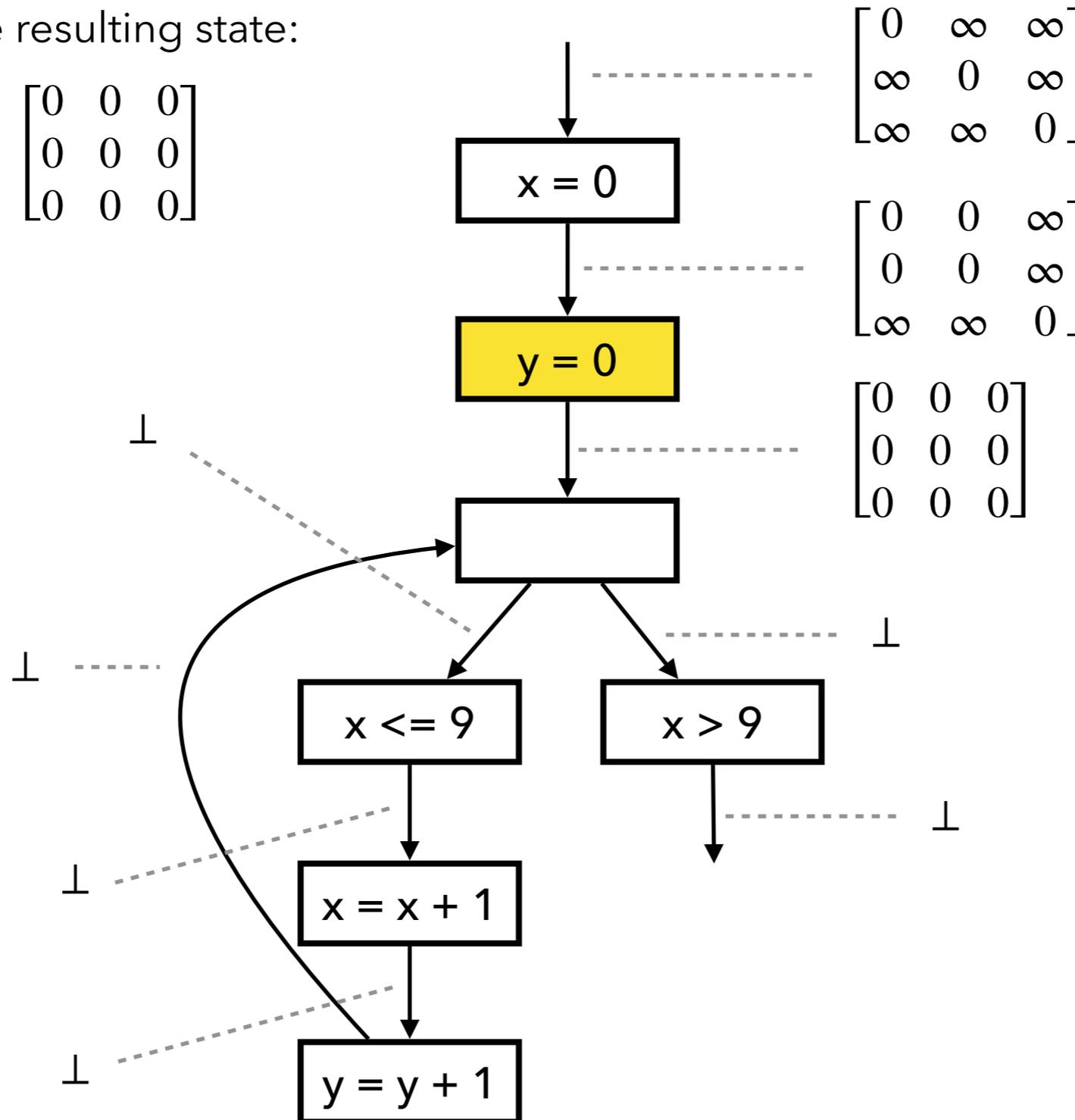
$$\begin{bmatrix} 0 & 0 & \infty \\ 0 & 0 & \infty \\ \infty & \infty & \infty \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \infty \\ 0 & \infty & \infty \end{bmatrix}$$



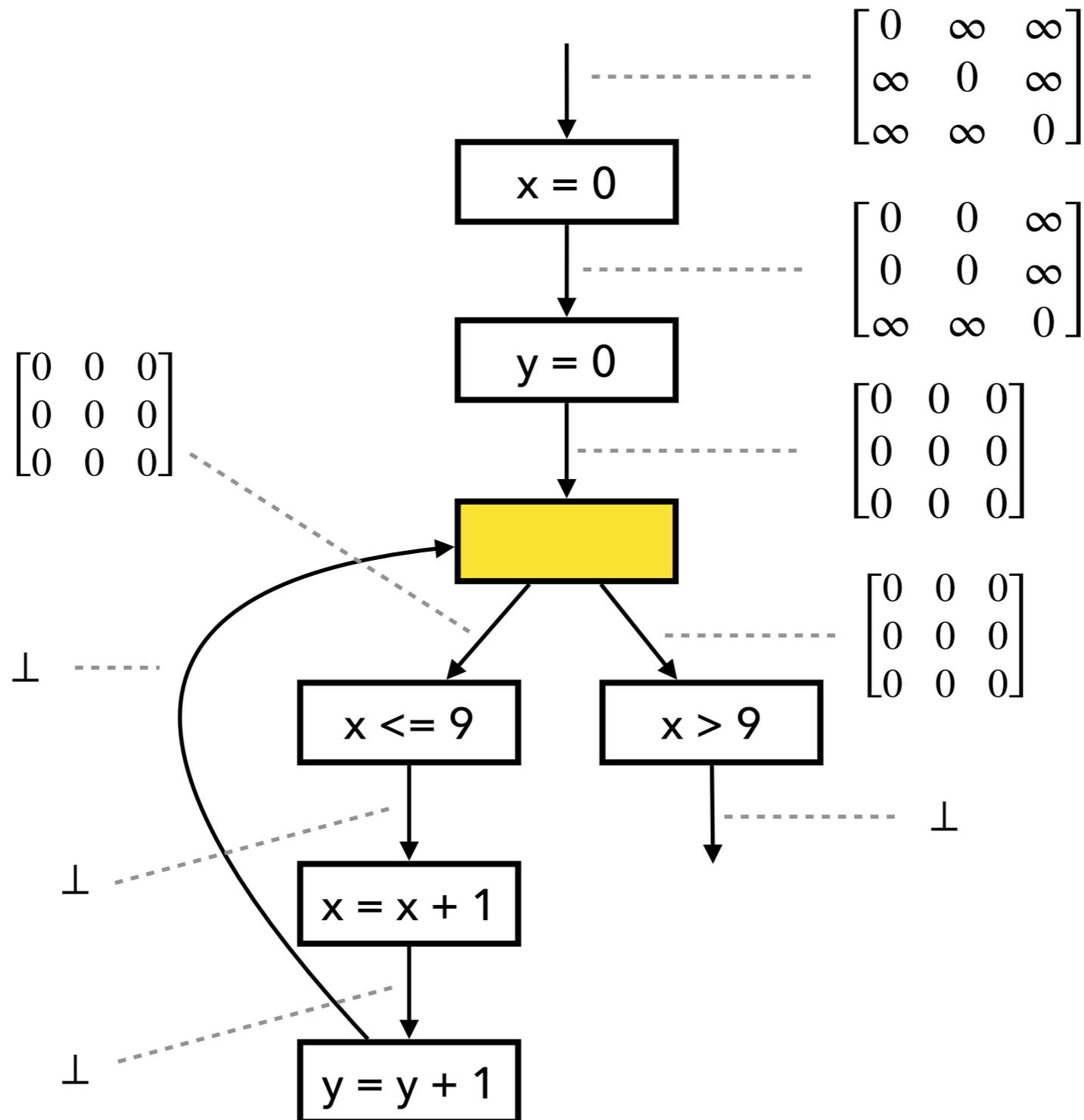
Fixed Point Comp. with Widening

3. Normalize the resulting state:

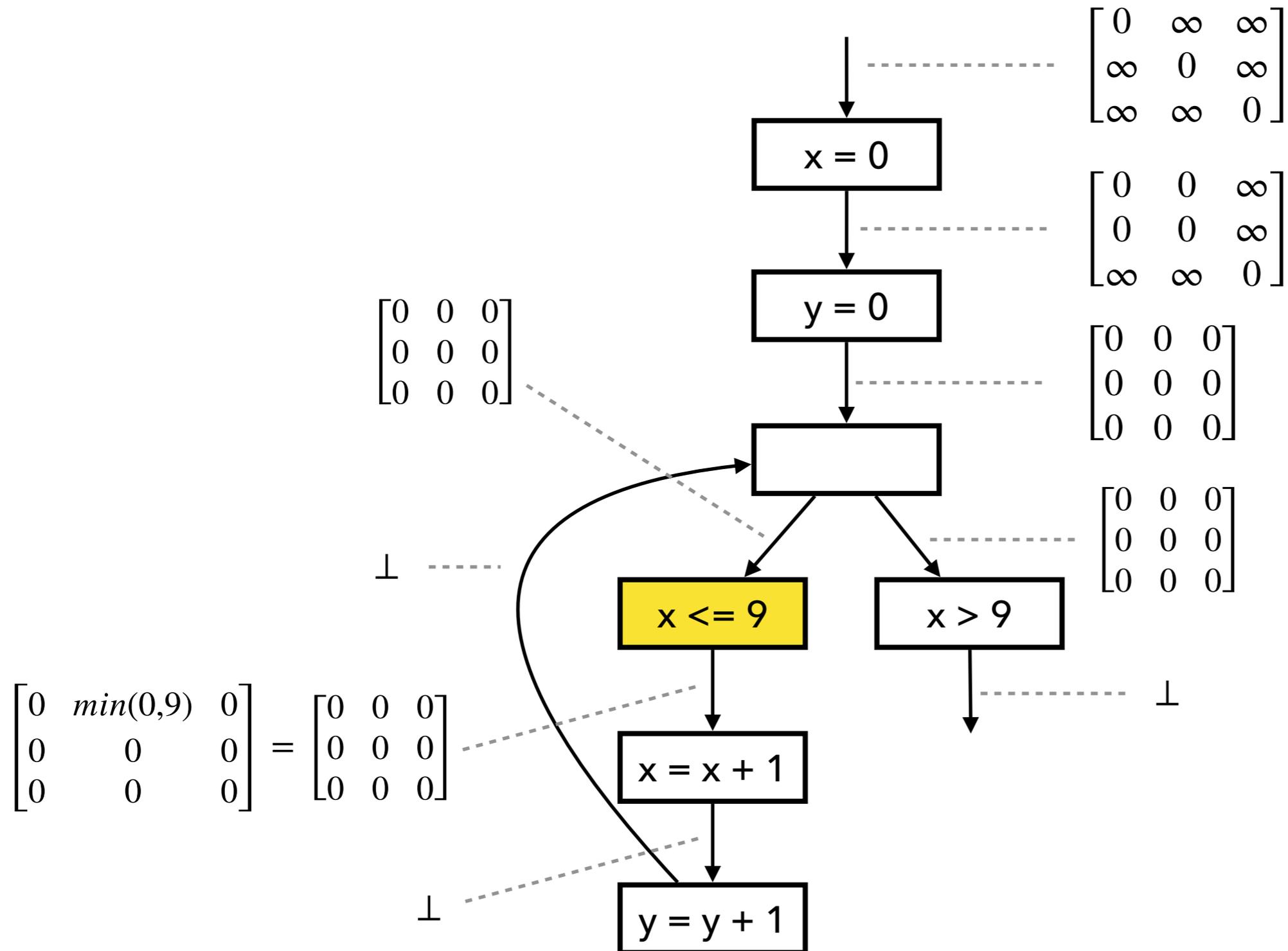
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \infty \\ 0 & \infty & \infty \end{bmatrix}^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



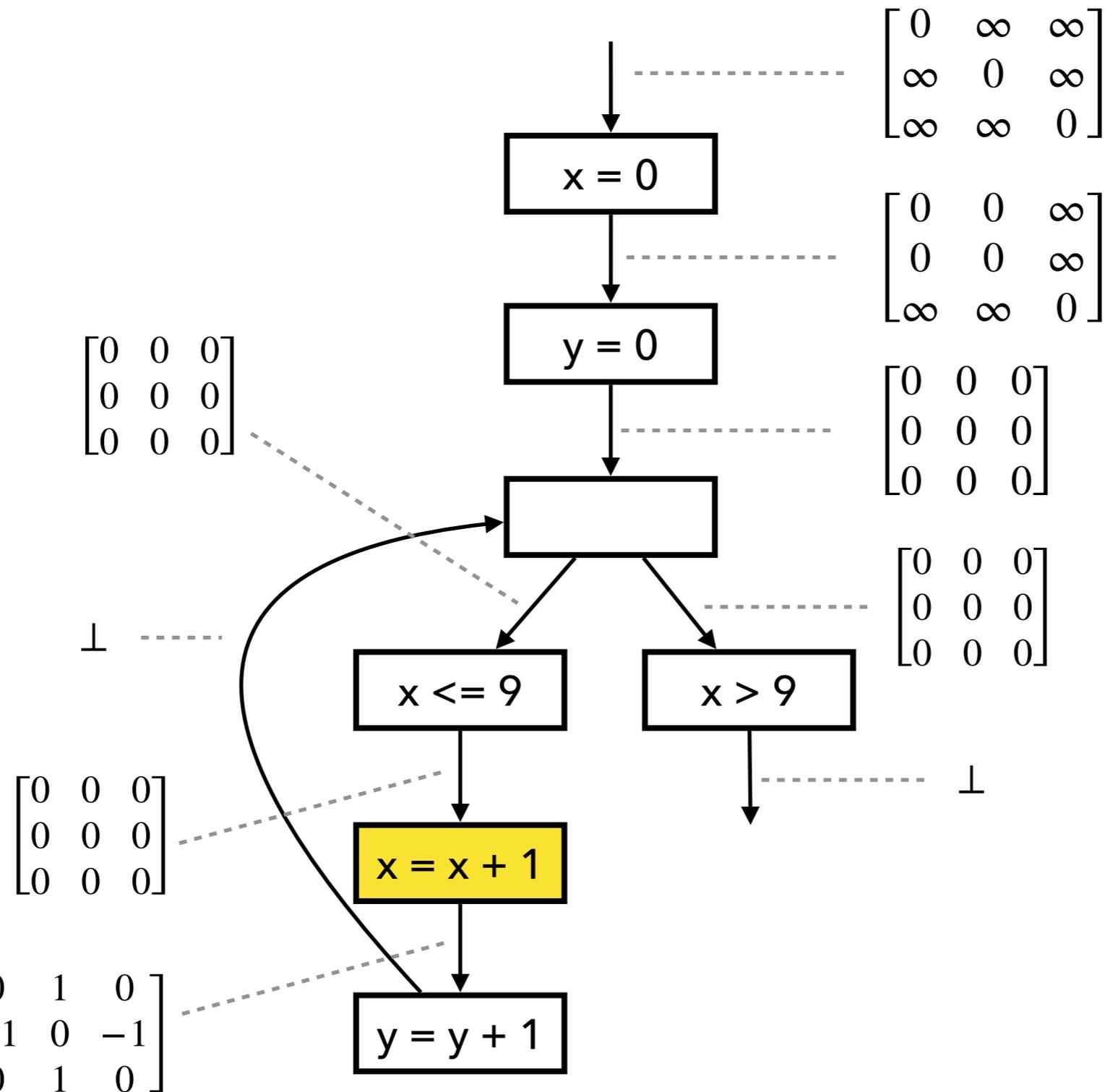
Fixed Point Comp. with Widening



Fixed Point Comp. with Widening



Fixed Point Comp. with Widening

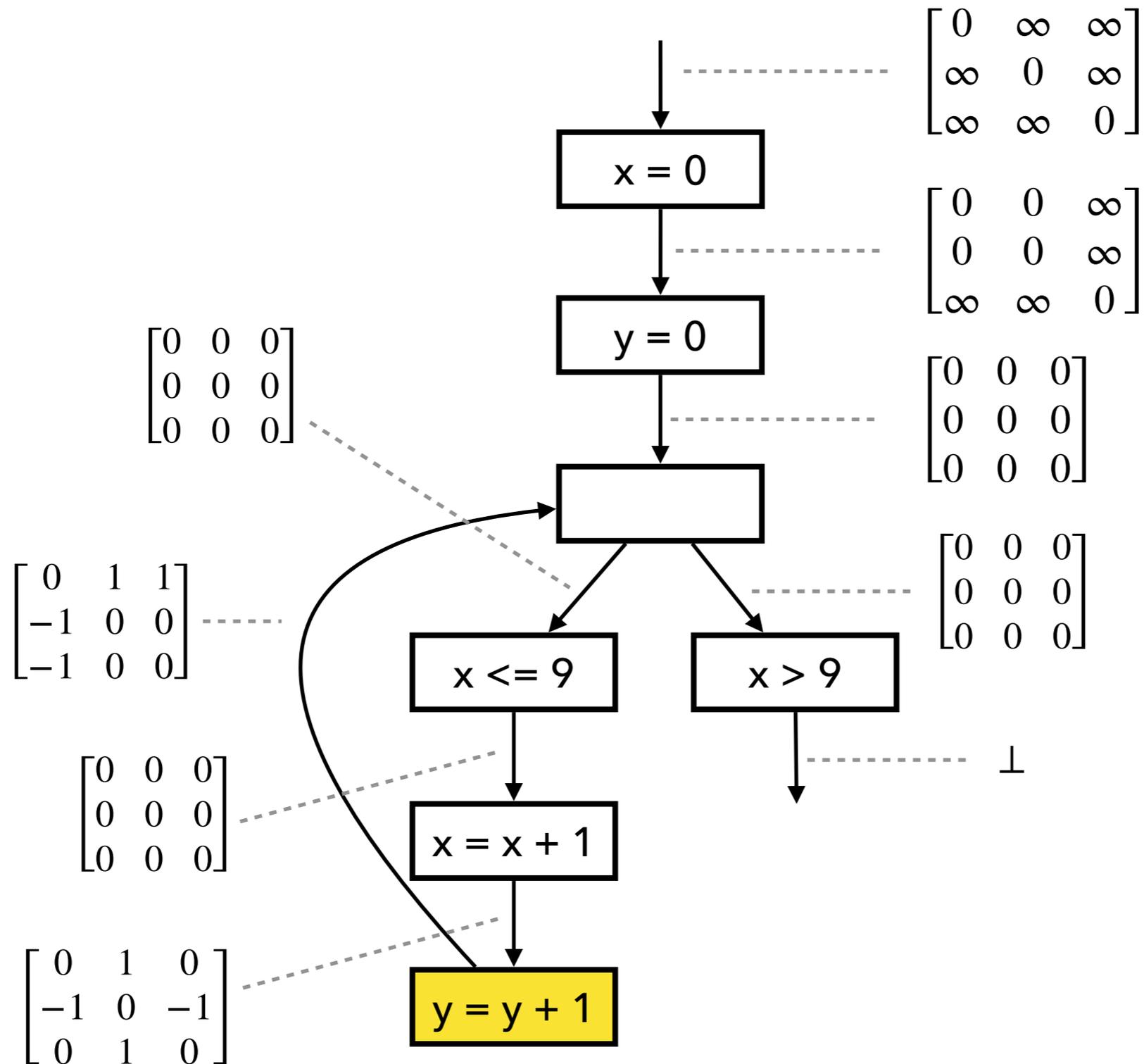


$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$x - x' \leq c \rightarrow x - x' \leq c + 1$$

$$x' - x \leq c \rightarrow x' - x \leq c - 1$$

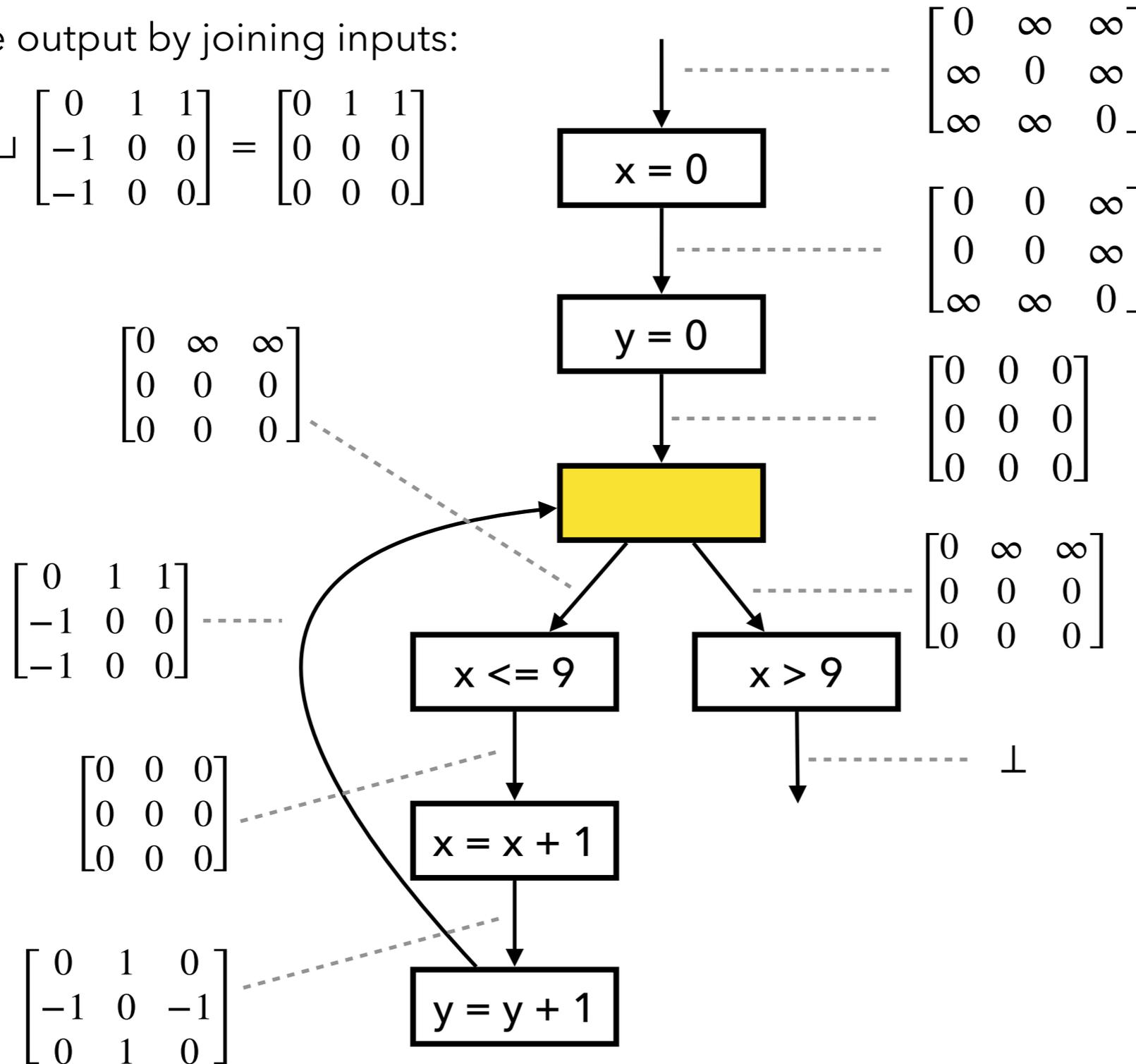
Fixed Point Comp. with Widening



Fixed Point Comp. with Widening

1. Compute output by joining inputs:

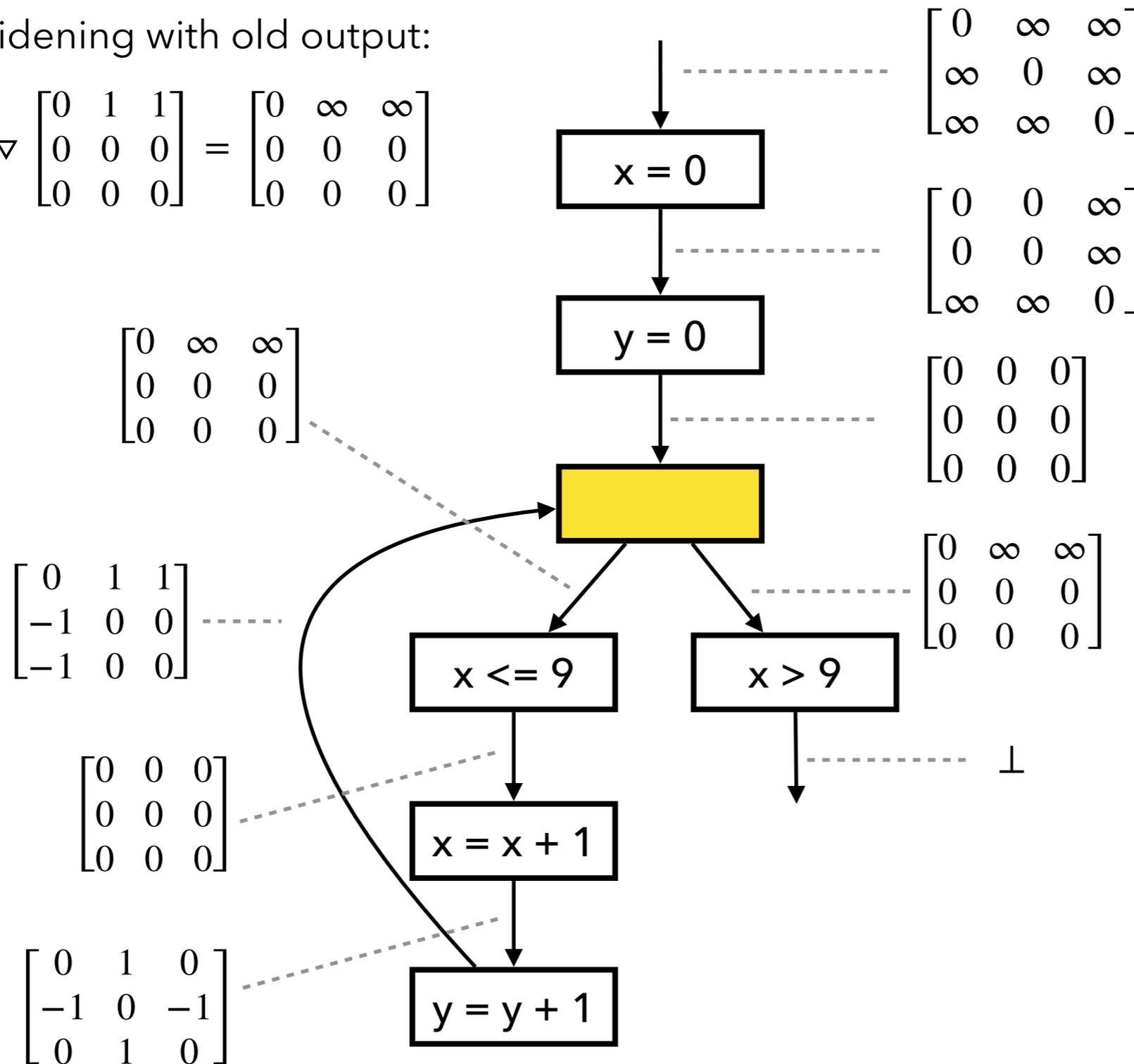
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sqcup \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Fixed Point Comp. with Widening

2. Apply widening with old output:

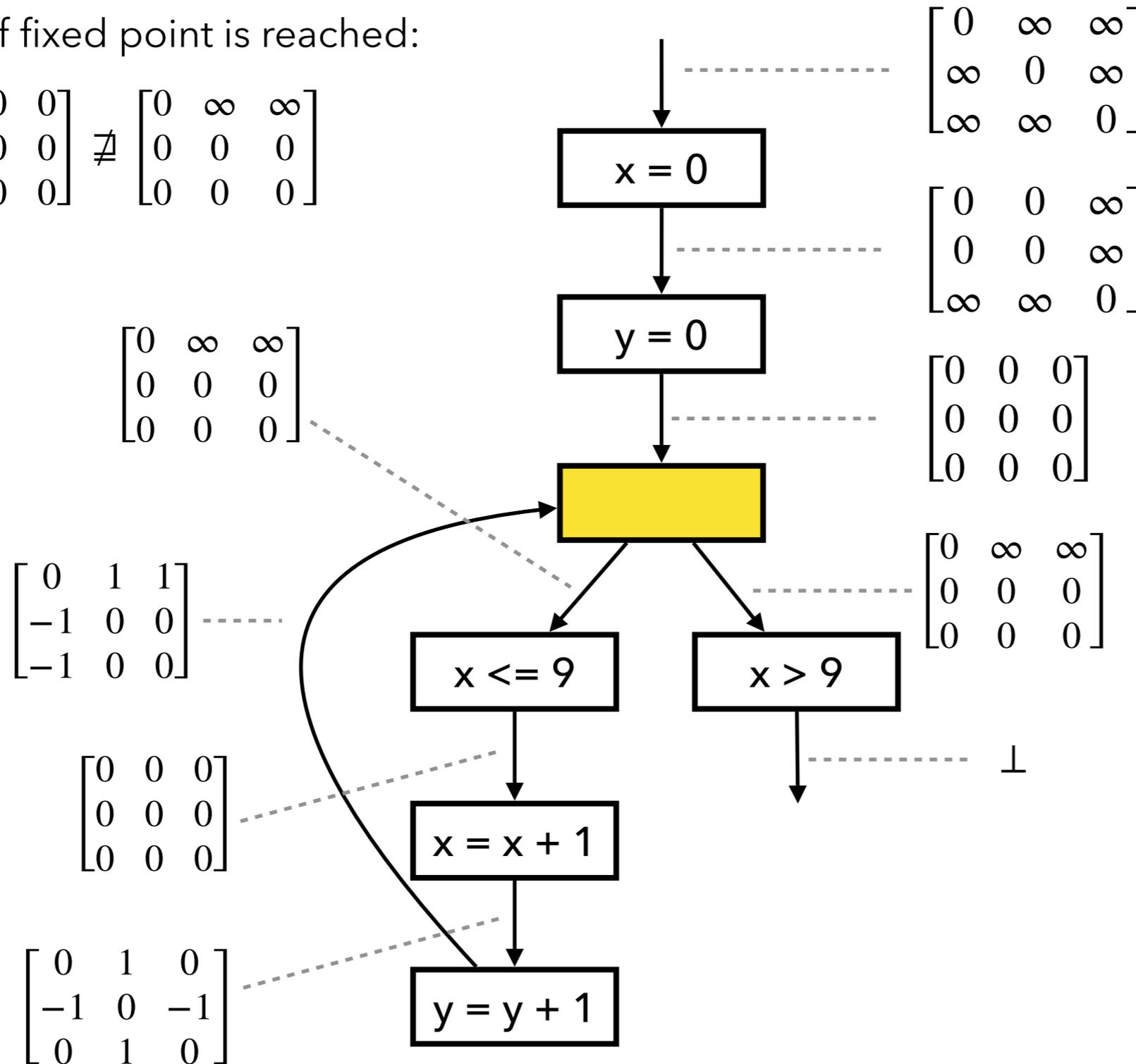
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \nabla \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \infty & \infty \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Fixed Point Comp. with Widening

3. Check if fixed point is reached:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & \infty & \infty \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Fixed Point Comp. with Widening

1. Add constraint "x ≤ 9":

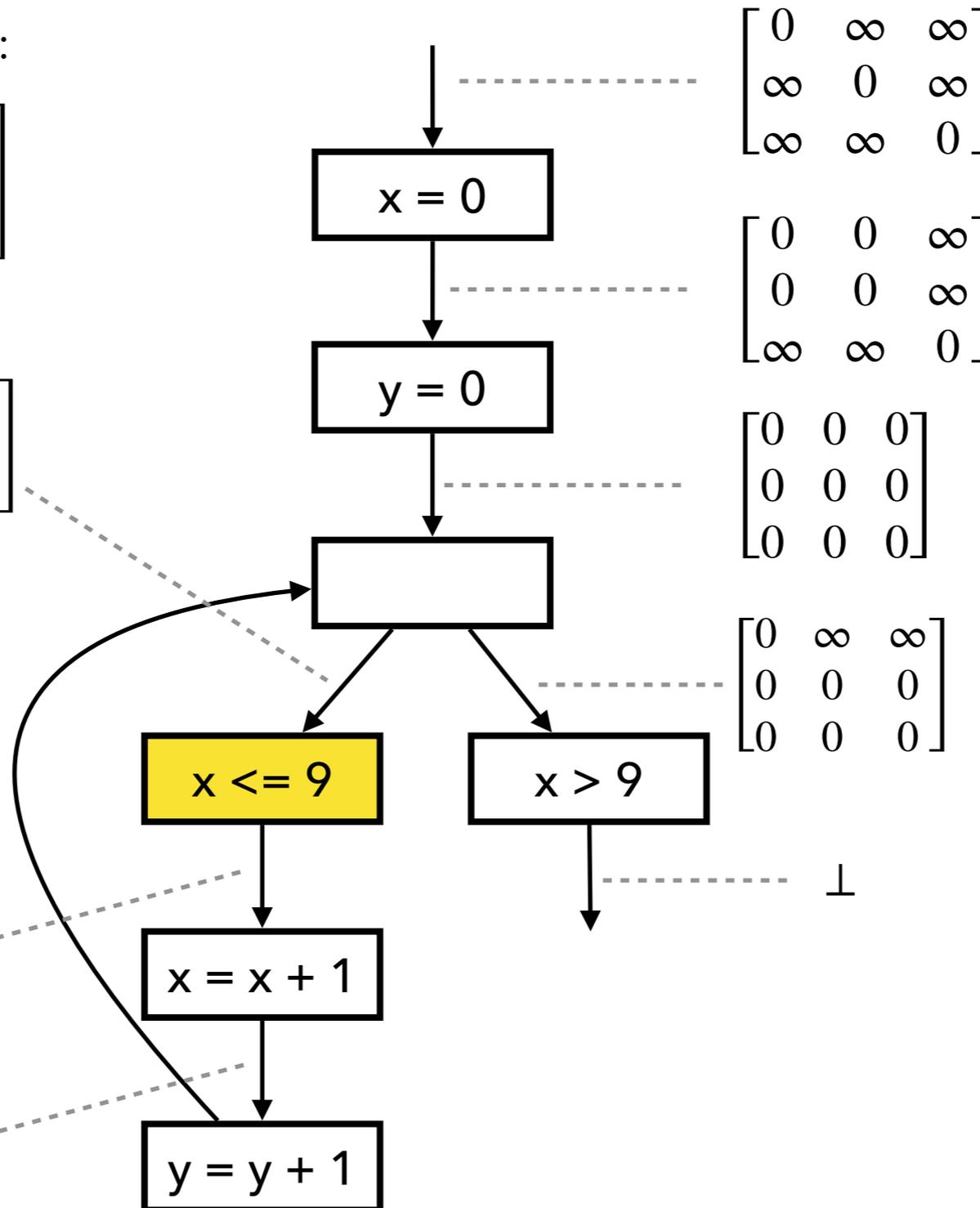
$$\begin{bmatrix} 0 & \infty & \infty \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 9 & \infty \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \infty & \infty \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 9 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

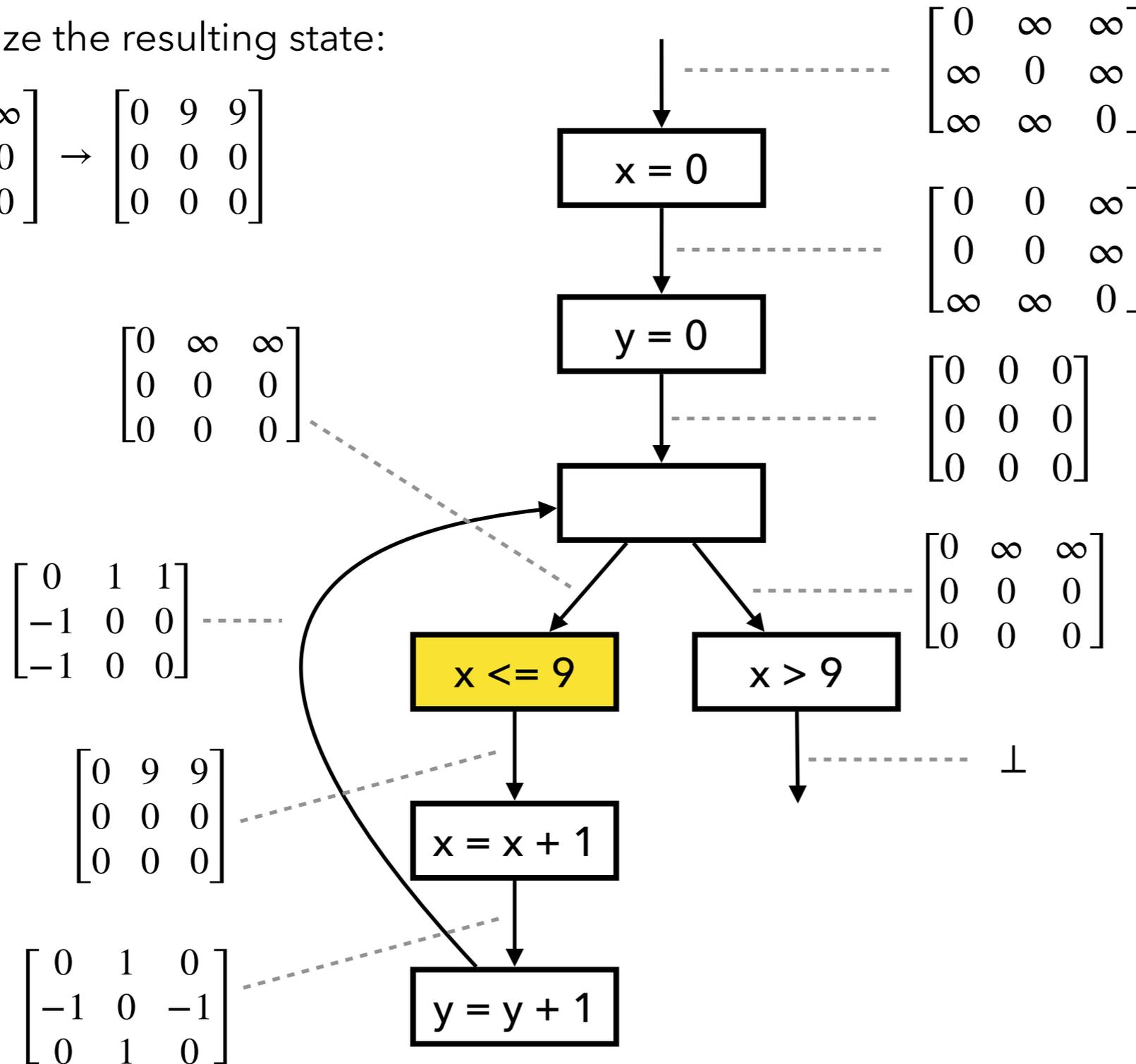
$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$



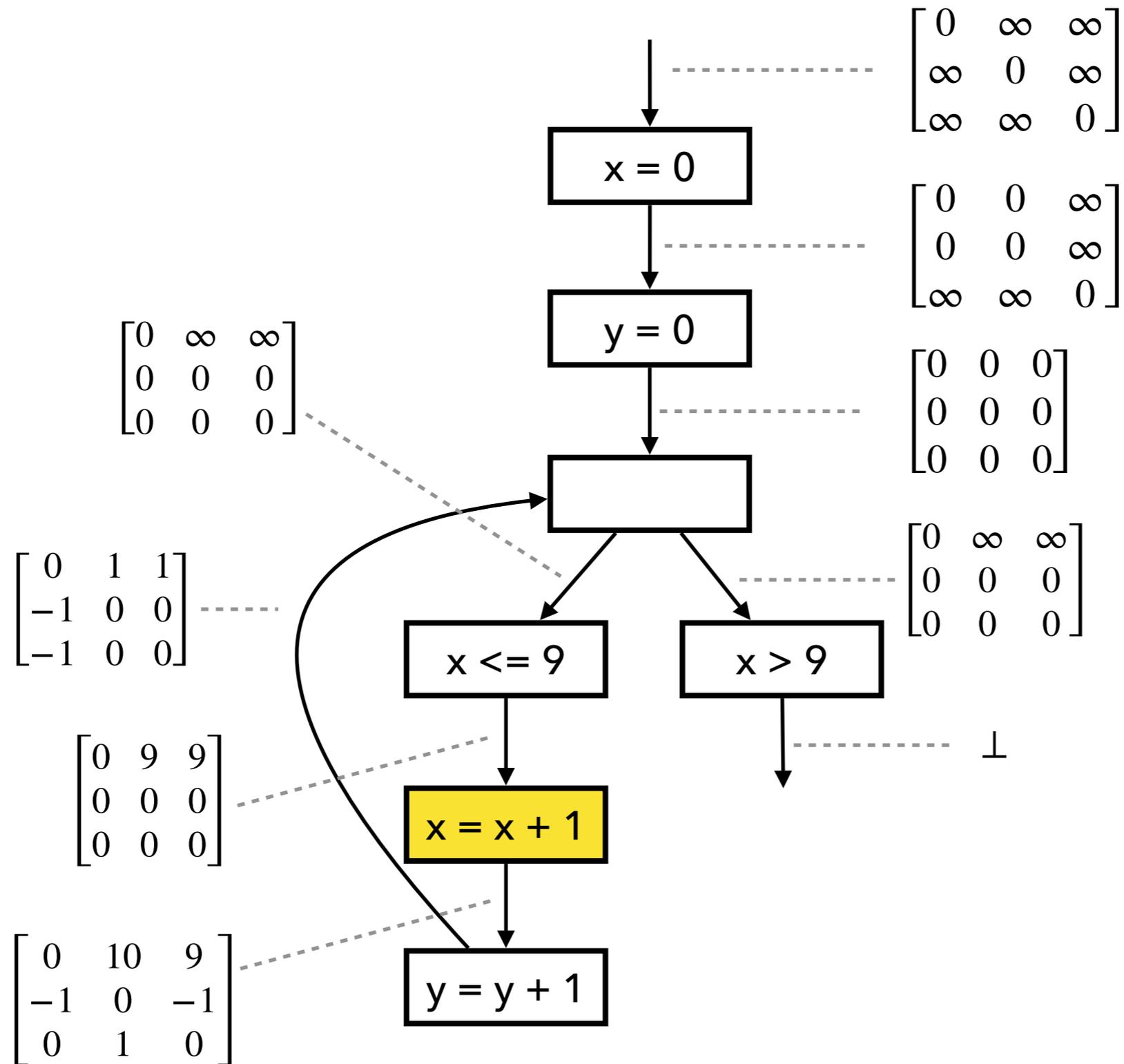
Fixed Point Comp. with Widening

2. Normalize the resulting state:

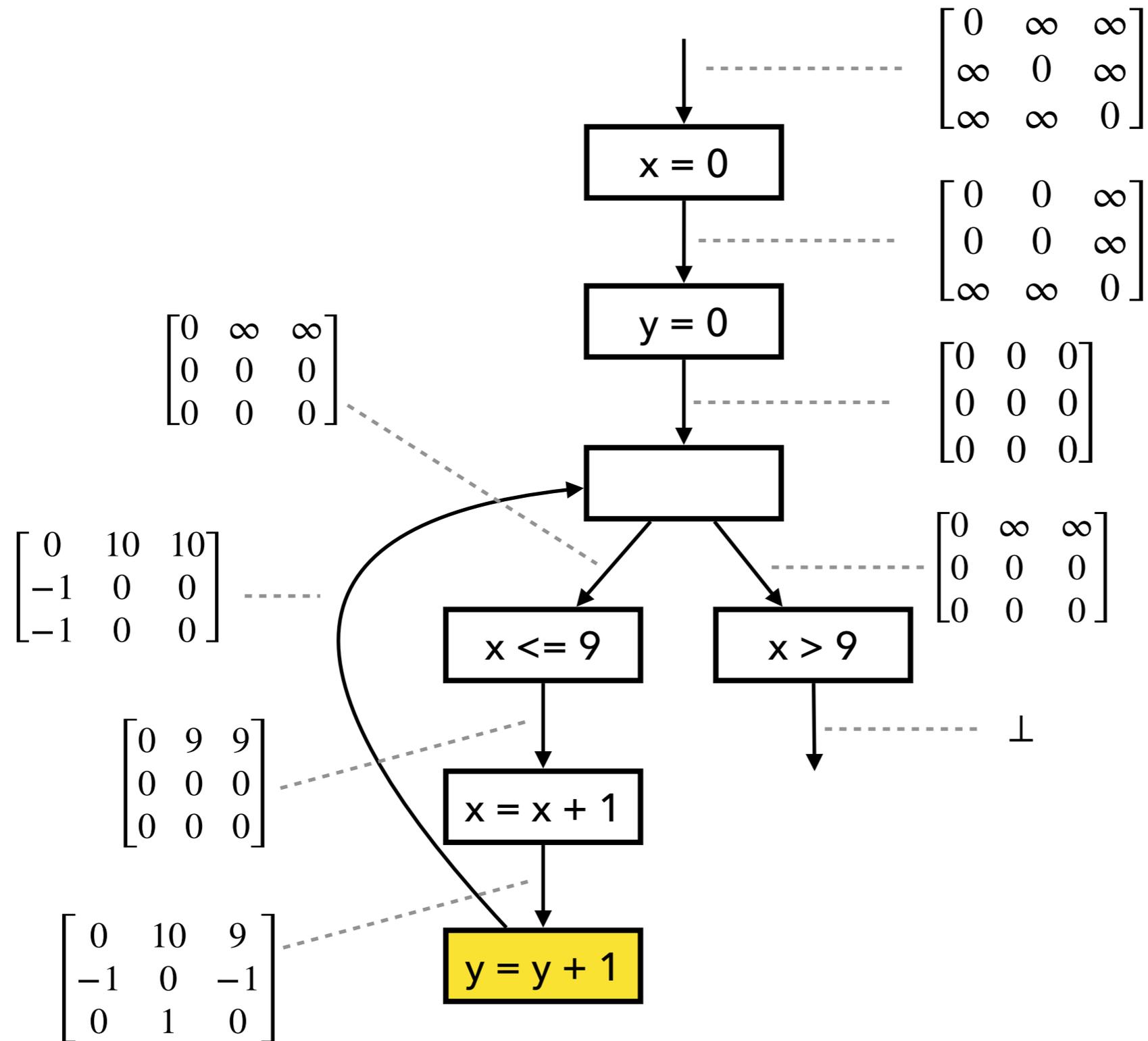
$$\begin{bmatrix} 0 & 9 & \infty \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 9 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Fixed Point Comp. with Widening



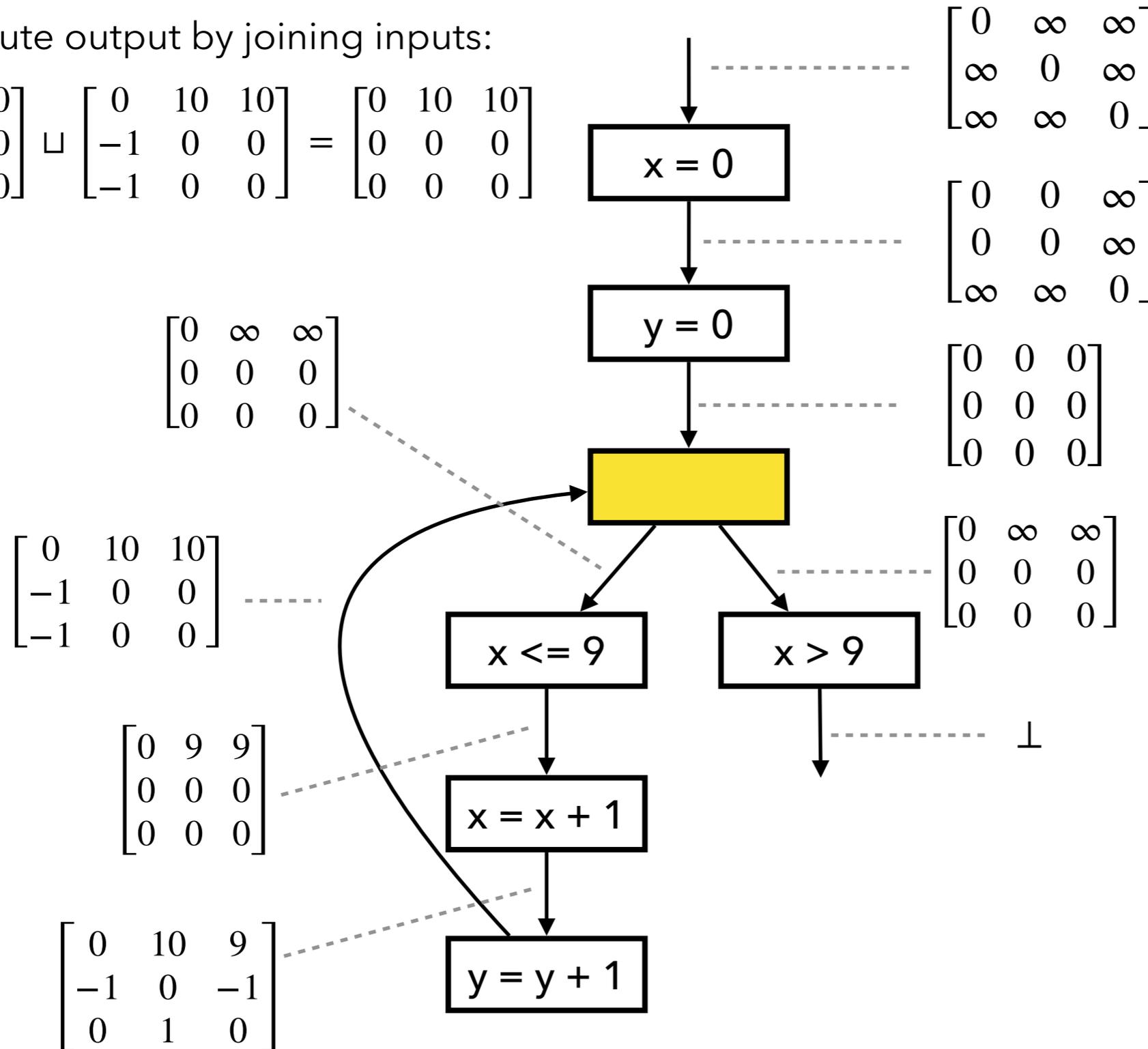
Fixed Point Comp. with Widening



Fixed Point Comp. with Widening

1. Compute output by joining inputs:

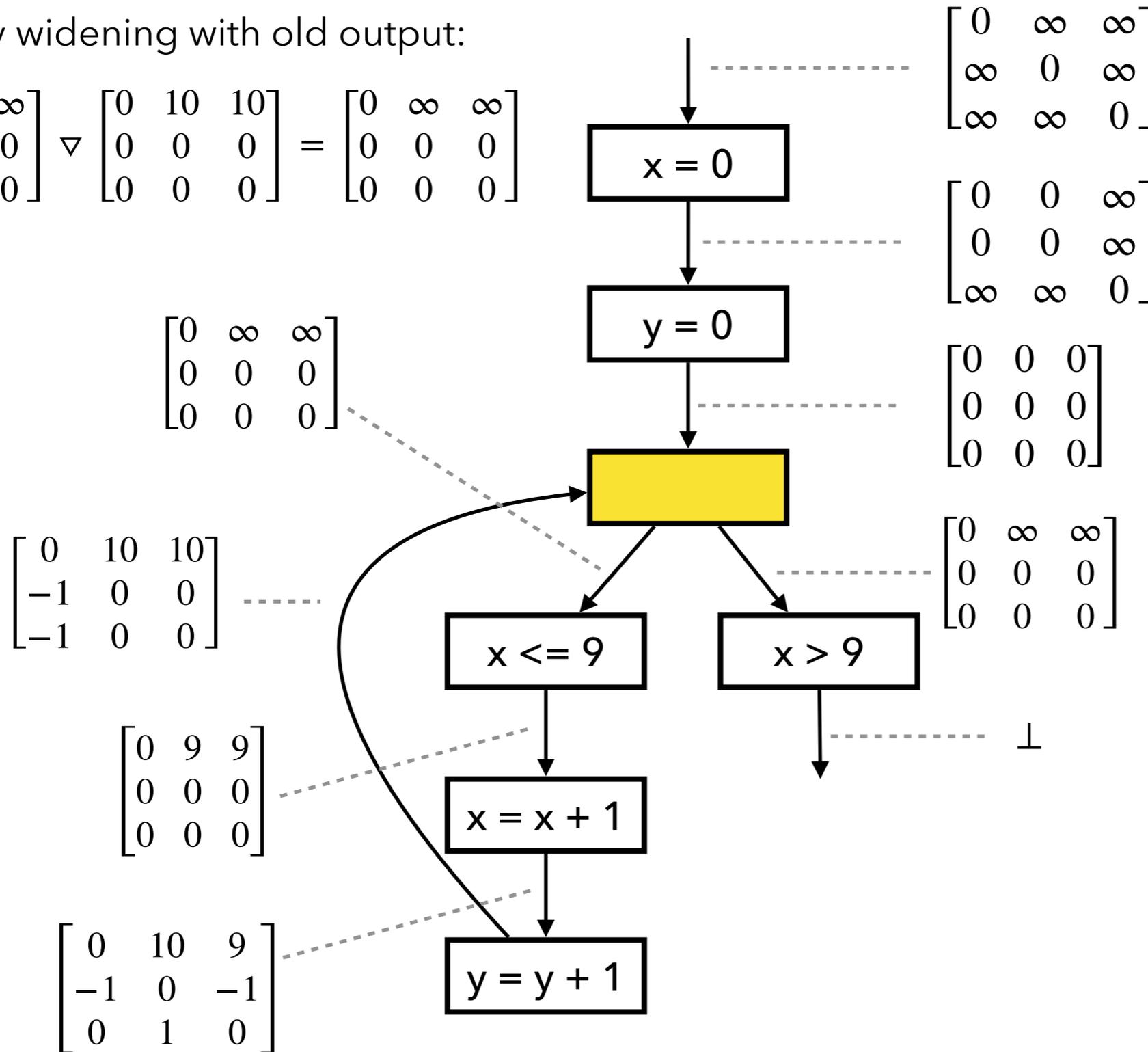
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sqcup \begin{bmatrix} 0 & 10 & 10 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 10 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Fixed Point Comp. with Widening

2. Apply widening with old output:

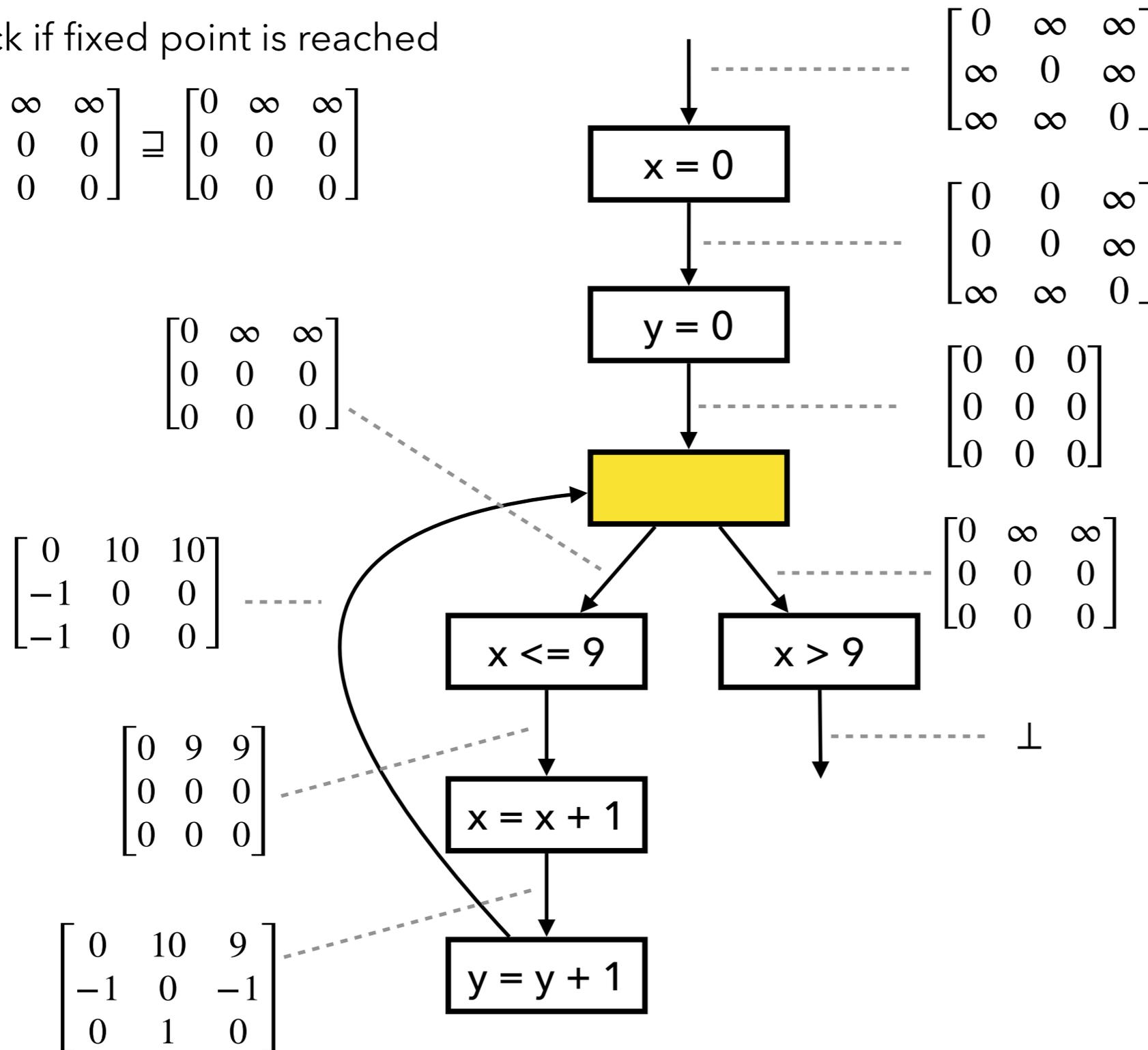
$$\begin{bmatrix} 0 & \infty & \infty \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \nabla \begin{bmatrix} 0 & 10 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \infty & \infty \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Fixed Point Comp. with Widening

3. Check if fixed point is reached

$$\begin{bmatrix} 0 & \infty & \infty \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sqsupseteq \begin{bmatrix} 0 & \infty & \infty \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Fixed Point Comp. with Widening

1. Add constraint "x > 9"

$$x > 9 \iff 0 - x \leq -10$$

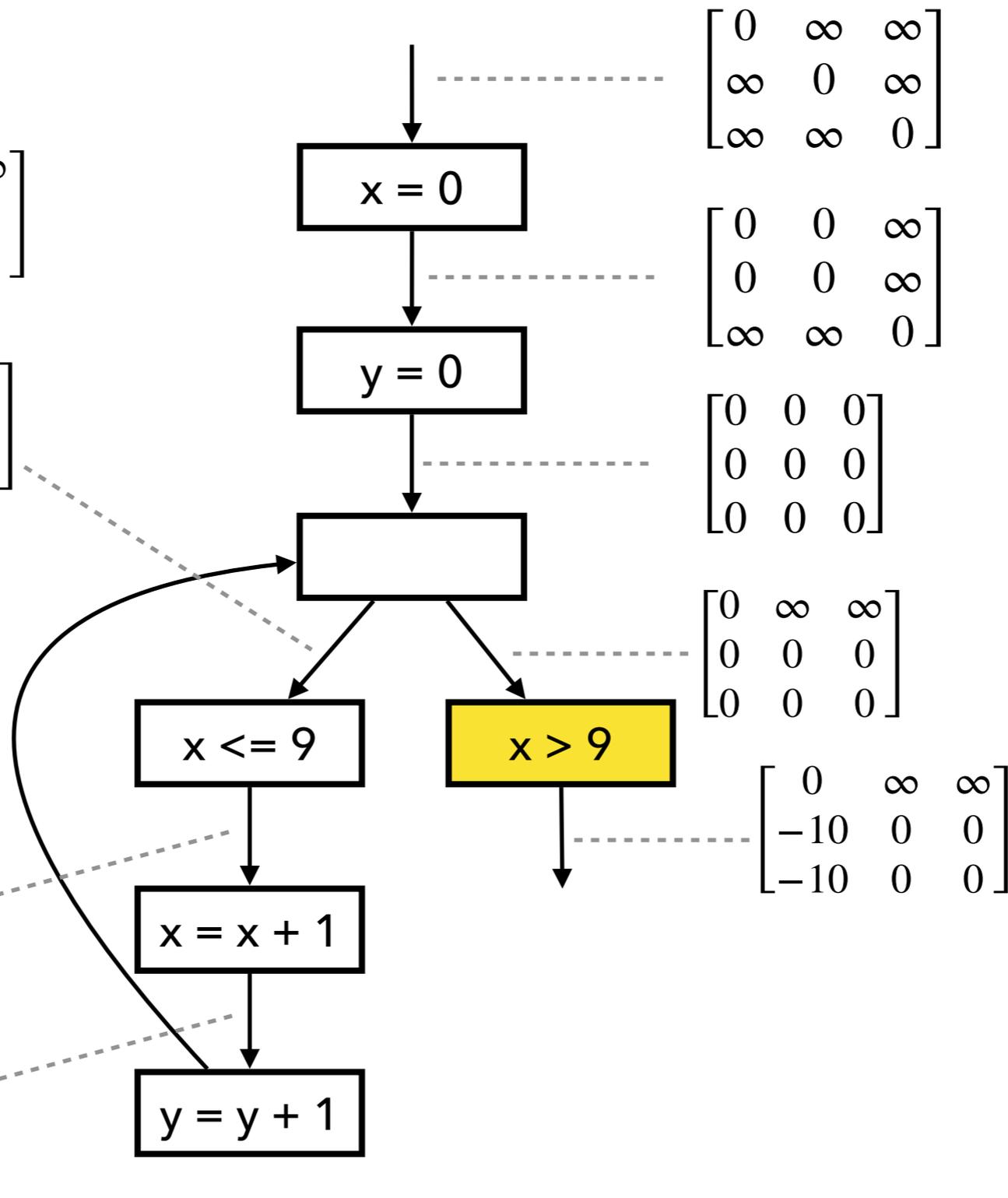
$$\begin{bmatrix} 0 & \infty & \infty \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & \infty & \infty \\ -10 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \infty & \infty \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 10 & 10 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 9 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

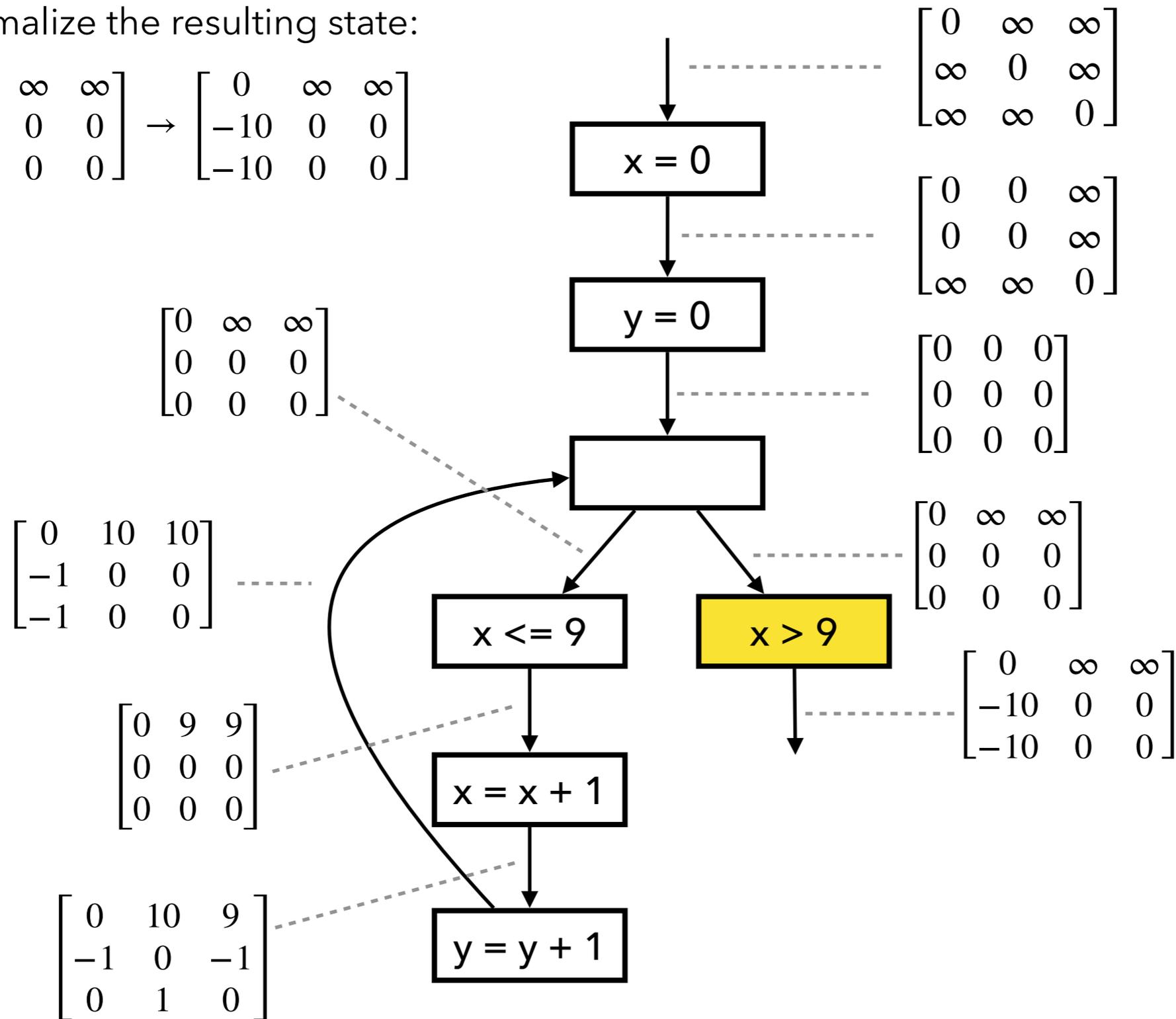
$$\begin{bmatrix} 0 & 10 & 9 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$



Fixed Point Comp. with Widening

2. Normalize the resulting state:

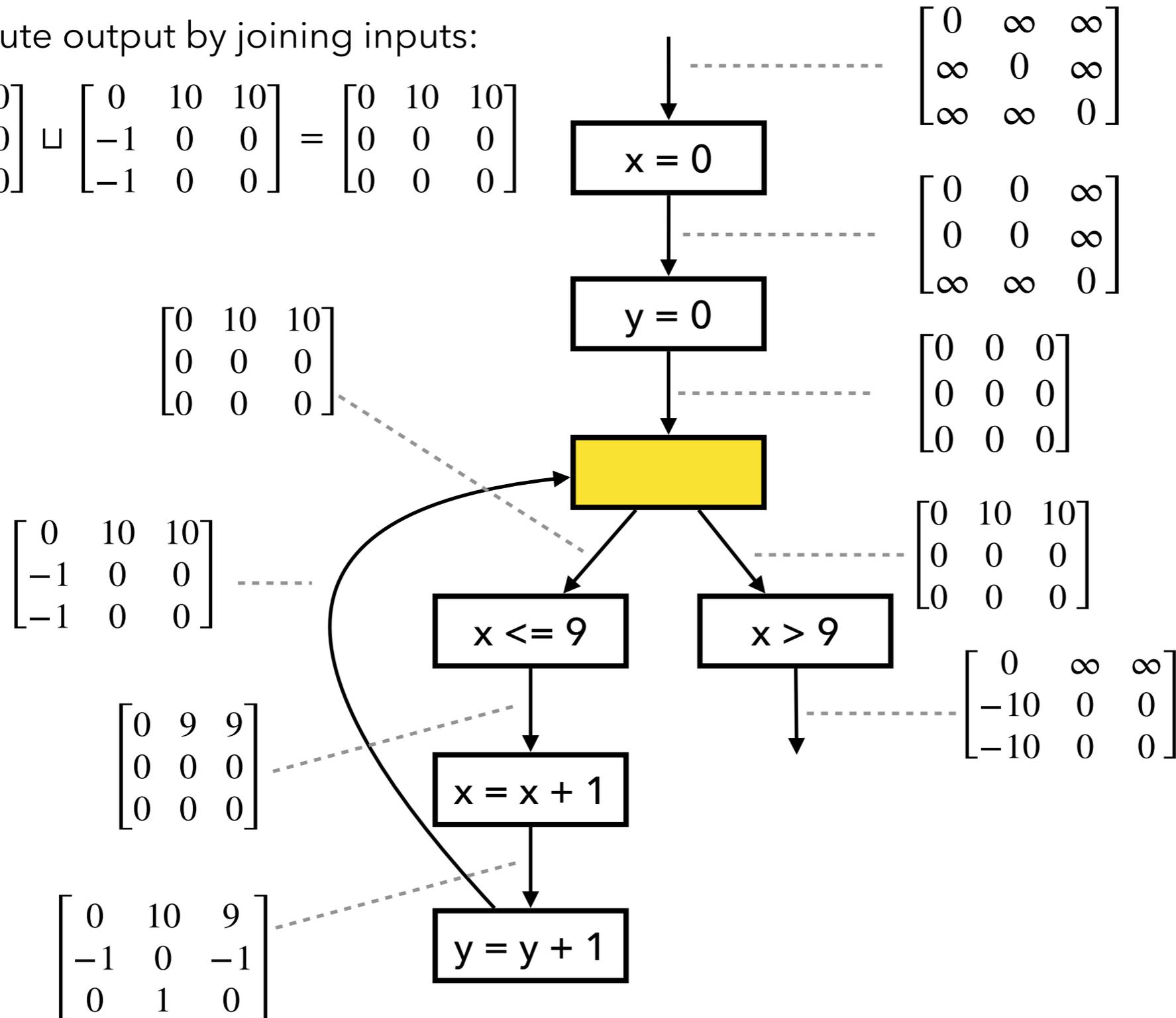
$$\begin{bmatrix} 0 & \infty & \infty \\ -10 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & \infty & \infty \\ -10 & 0 & 0 \\ -10 & 0 & 0 \end{bmatrix}$$



Fixed Point Comp. with Narrowing

1. Compute output by joining inputs:

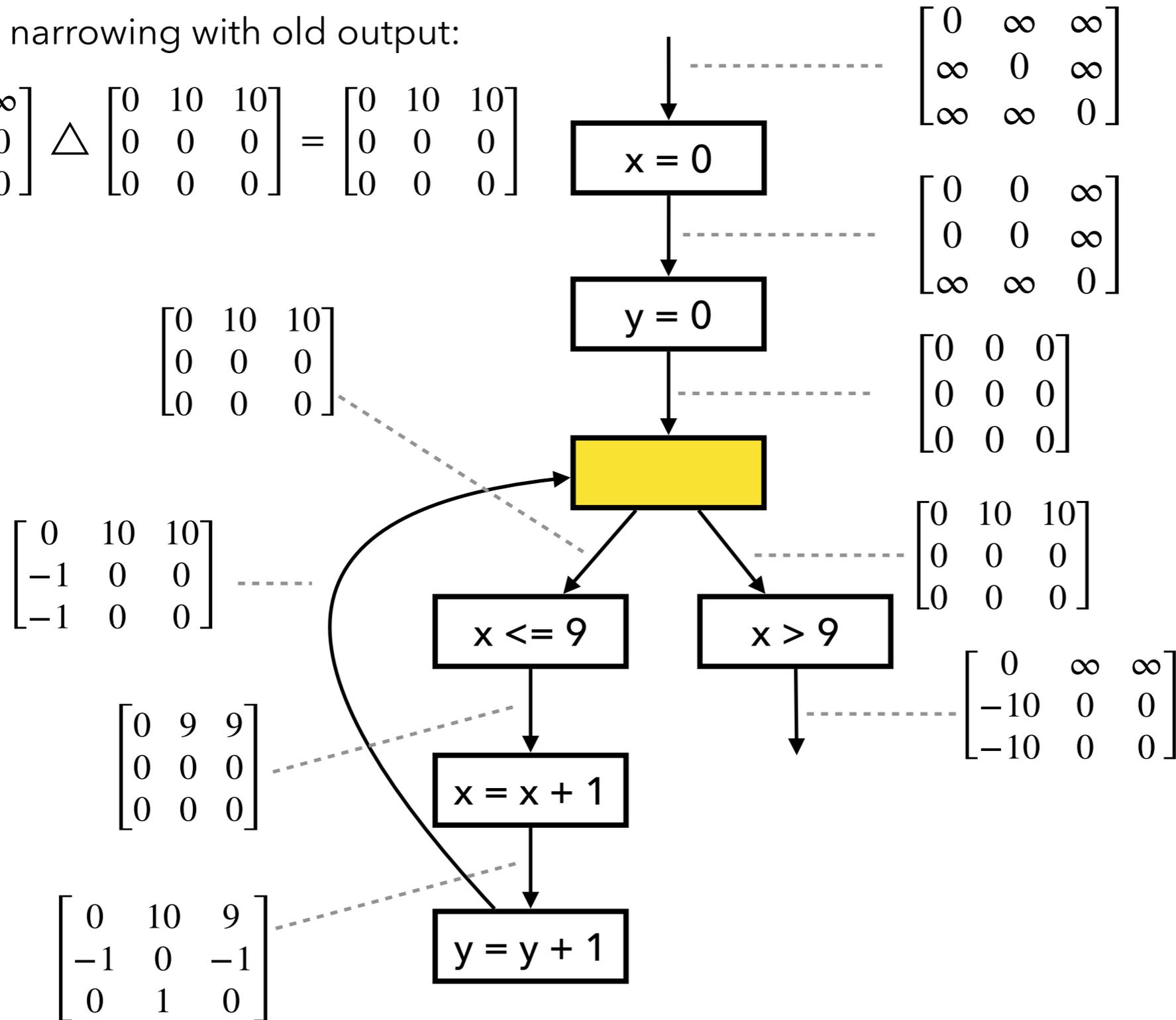
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sqcup \begin{bmatrix} 0 & 10 & 10 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 10 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Fixed Point Comp. with Narrowing

2. Apply narrowing with old output:

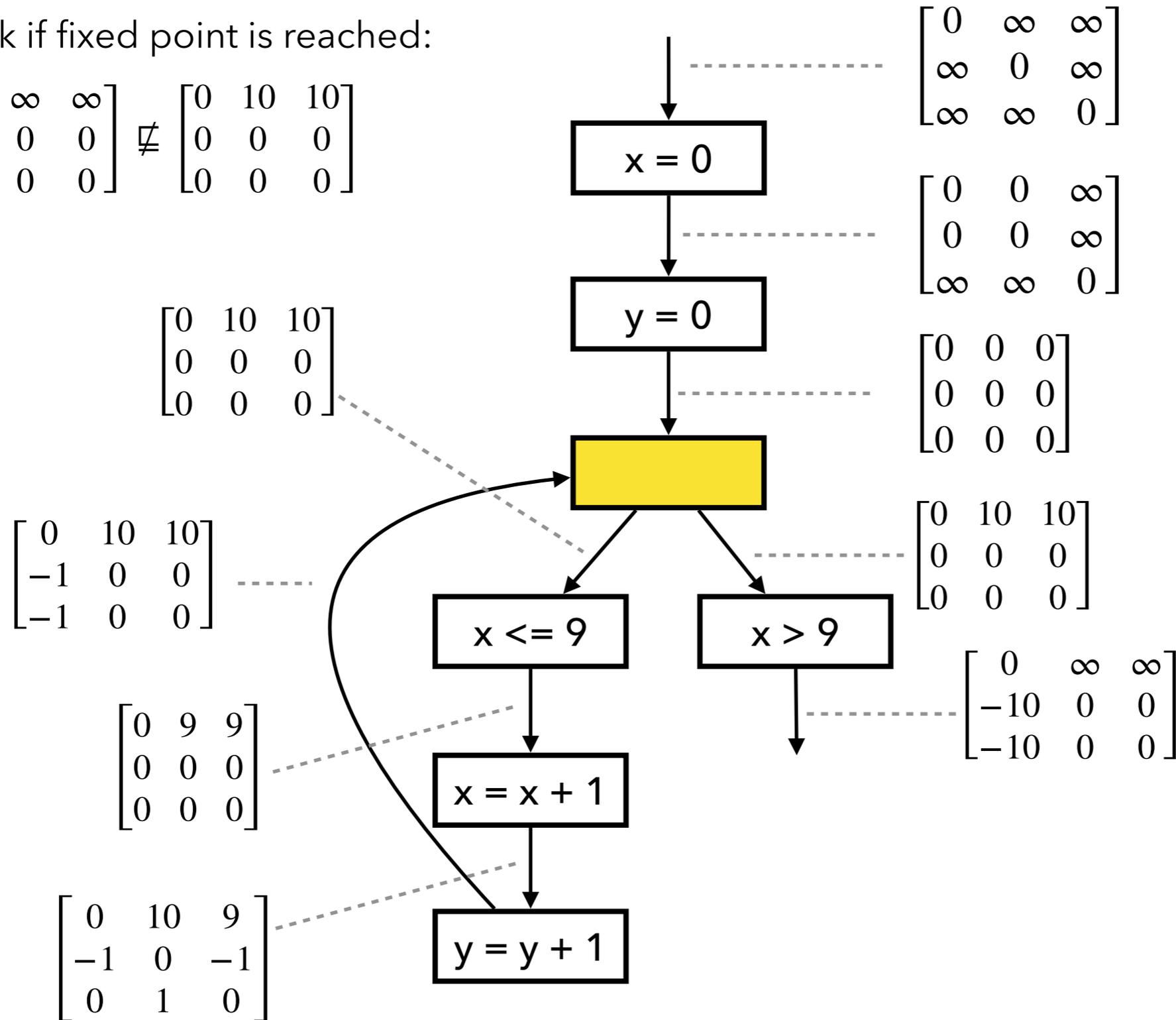
$$\begin{bmatrix} 0 & \infty & \infty \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \triangle \begin{bmatrix} 0 & 10 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 10 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



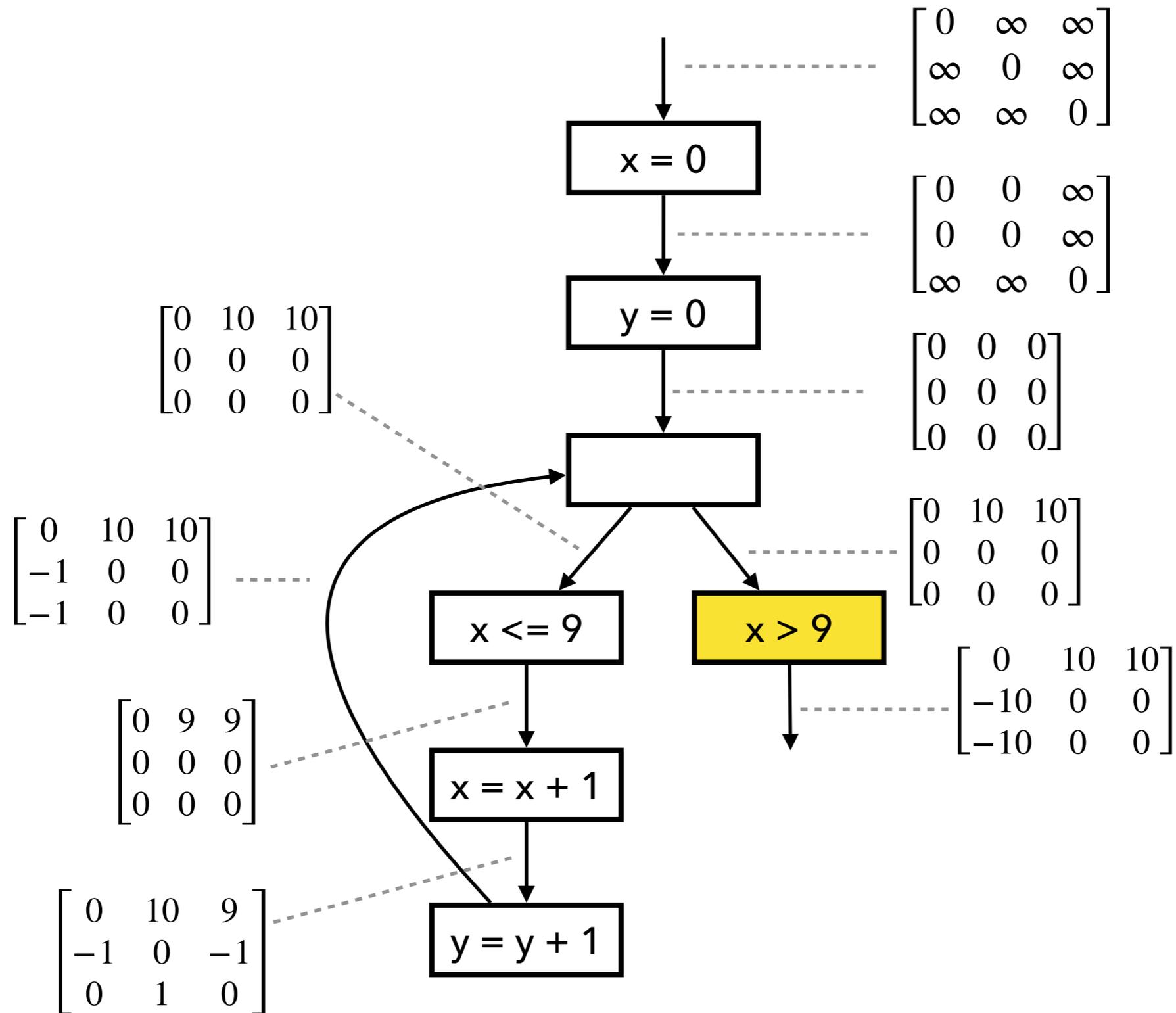
Fixed Point Comp. with Narrowing

3. Check if fixed point is reached:

$$\begin{bmatrix} 0 & \infty & \infty \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 10 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



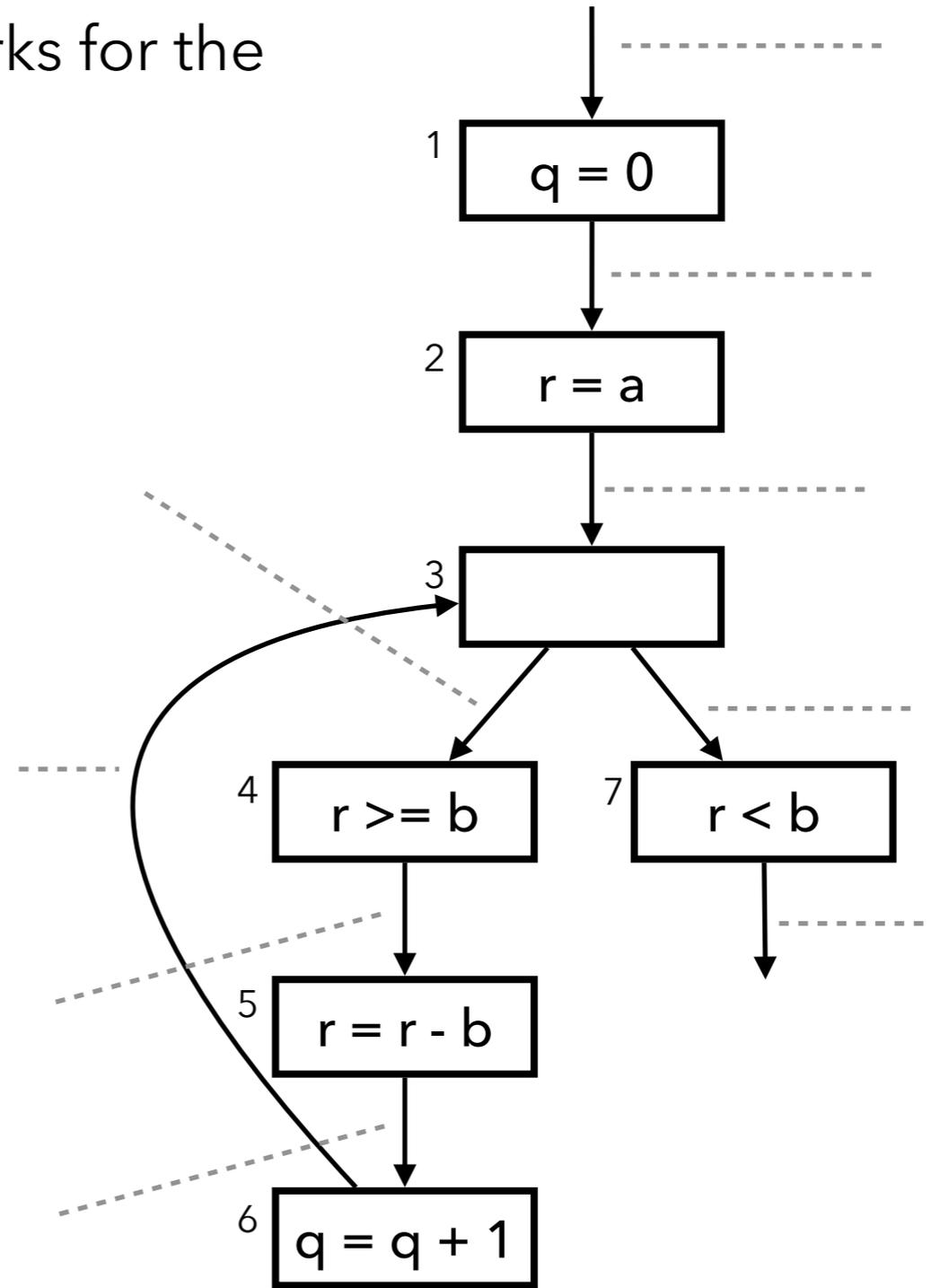
Fixed Point Comp. with Narrowing



Exercise

Describe how the zone analysis works for the following example.

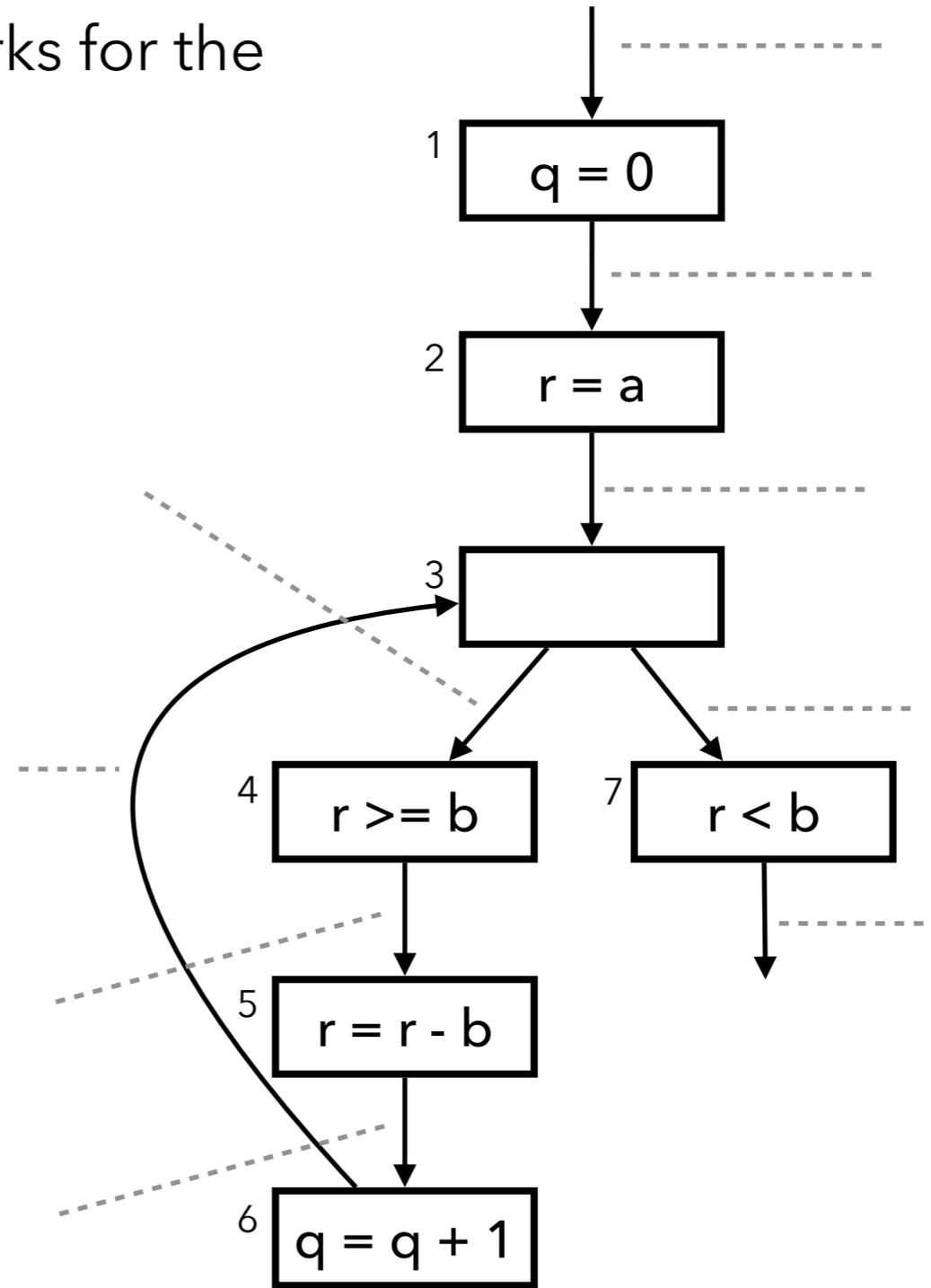
```
// a >= 0, b >= 0
q = 0;
r = a;
while (r >= b) {
    r = r - b;
    q = q + 1;
}
assert (q >= 0);
assert (r >= 0);
```



Exercise

Describe how the zone analysis works for the following example.

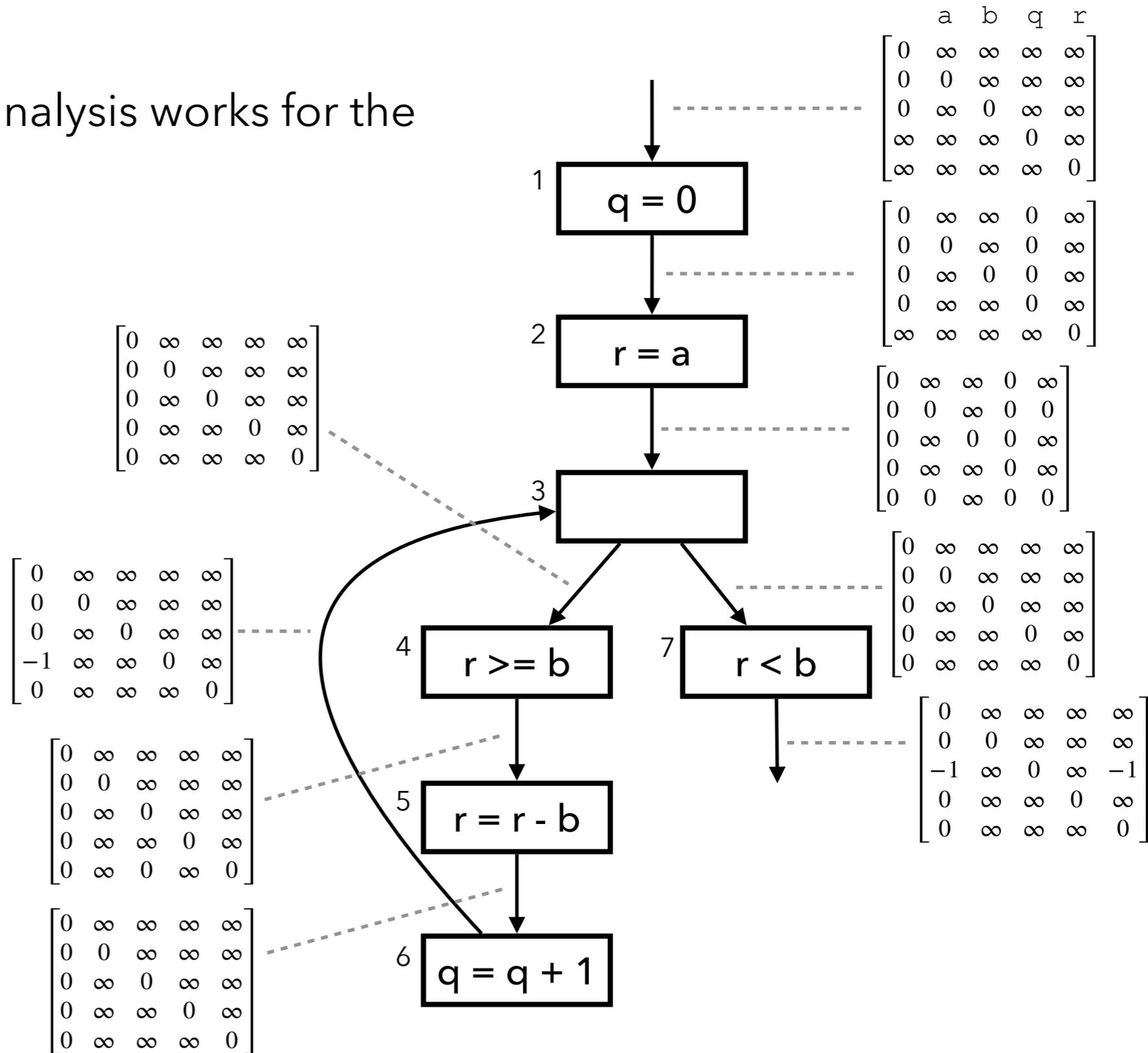
```
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q = 0;
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while (r >= b) {
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    q = q + 1;
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assert (q >= 0);
assert (r >= 0);
```



Exercise

Describe how the zone analysis works for the following example.

```
// a >= 0, b >= 0
q = 0;
r = a;
while (r >= b) {
    r = r - b;
    q = q + 1;
}
assert (q >= 0);
assert (r >= 0);
```



Numerical Abstract Domains

- Interval domain: e.g., $1 \leq x \leq 10$
- Octagon domain: e.g., $1 \leq x - y \leq 10$
- Polyhedra domain: e.g., $1 \leq 2x + y + z \leq 10$
- Congruence domain: e.g., $x \equiv 2 \pmod{4}$
- Disjunctive domain: e.g., $1 \leq x \leq 10 \vee 20 \leq x$
- ...