COSE312: Compilers

Lecture 5 — Syntax Analysis (3): Bottom-Up Parsing

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Expression Grammar

Expression grammar:

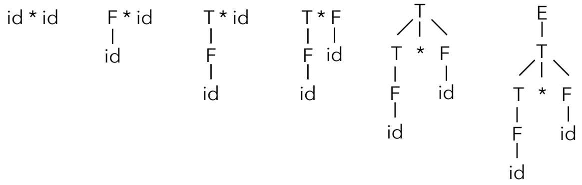
$$E \rightarrow E + E \mid E * E \mid (E) \mid \mathbf{id}$$

Unambiguous version:

$$\begin{array}{llll} (1) & E & \rightarrow & E + T \\ (2) & E & \rightarrow & T \\ (3) & T & \rightarrow & T * F \\ (4) & T & \rightarrow & F \\ (5) & F & \rightarrow & (E) \\ (6) & F & \rightarrow & \mathbf{id} \end{array}$$

Bottom-Up Parsing

- Construct a parse tree beginning at the leaves and working up towards the root.
- Ex) for input $\mathbf{id} * \mathbf{id}$:



- A process of "reducing" a string w to the start symbol.
- Construct the rightmost-derivation in reverse:

$$E \Rightarrow T \Rightarrow T * F \Rightarrow T * \mathbf{id} \Rightarrow F * \mathbf{id} \Rightarrow \mathbf{id} * \mathbf{id}$$

Handle

- In bottom-up parsing, we have to make decisions about when to reduce and what production to apply.
- For instance, for $T * \mathbf{id}$, we reduce $\mathbf{id}$ to F because reducing T does not lead to a right-sentential form.
- Handle: a substring that matches the body of a production and whose reduction leads to a right-sentential form.
- A bottom-up parsing is a process of finding a handle and reducing it.

Right Sentential Form	Handle	Reducing Production
$\mathbf{id}_1 * \mathbf{id}_2$	$\mathbf{id}_1$	$F \to \mathbf{id}$
$F * \mathbf{id}_2$	F	$T \to F$
$T * \mathbf{id}_2$	$\mathbf{id}_2$	$F \to \mathbf{id}$
$T * F$	$T * F$	$T \to T * F$
T	T	$E \to T$

LR Parsing

- The most prevalent type of bottom-up parsing.
- Handles are recognized by a deterministic finite automaton.
- LR(k)
  - "L": Left-to-right scanning of the input
  - "R": Rightmost-derivation in reverse
  - "k": k-tokens lookahead
- We consider LR(0), SLR, LR(1), LALR(1) parsing algorithms.

Why LR parsing?

- Widely used:
  - Most automatic parser generators are based on LR parsing
- General and powerful:
  - LL(k) $\subseteq$ LR(k)
  - Many programming languages can be described by LR grammars

LR Parsing Overview

An LR parser has a *stack* and *an input*. Based on the lookahead and stack contents, perform two kinds of actions:

- Shift
  - performed when the top of the stack is not a handle
  - move the first input token to the stack
- Reduce
  - performed when the top of the stack is a handle
  - choose a rule $X \rightarrow A\ B\ C$; pop C, B, A; push X

Example: $\mathbf{id} * \mathbf{id}$

$$
\begin{array}{llll}
(1) & E & \rightarrow & E + T \\
(2) & E & \rightarrow & T \\
(3) & T & \rightarrow & T * F \\
(4) & T & \rightarrow & F \\
(5) & F & \rightarrow & (E) \\
(6) & F & \rightarrow & \mathbf{id}
\end{array}
$$

Stack	Input	Action
$\$$	$\mathbf{id} * \mathbf{id}\$$	shift
$\$\mathbf{id}$	$*\mathbf{id}\$$	reduce by $F \rightarrow \mathbf{id}$
$\$F$	$*\mathbf{id}\$$	reduce by $T \rightarrow F$
$\$T$	$*\mathbf{id}\$$	shift
$\$T*$	$\mathbf{id}\$$	shift
$\$T * \mathbf{id}$	$\$$	reduce by $F \rightarrow \mathbf{id}$
$\$T * F$	$\$$	reduce by $T \rightarrow T * F$
$\$T$	$\$$	reduce by $E \rightarrow T$
$\$E$	$\$$	shift (accept)

Recognizing Handles

By using a deterministic finite automaton. The transition table (parsing table) for the expression grammar:

State	id	+	*	(	)	$	E	T	F
0	$s5$			$s4$			$g1$	$g2$	$g3$
1		$s6$				acc			
2		$r2$	$s7$		$r2$	$r2$			
3		$r4$	$r4$		$r4$	$r4$			
4	$s5$			$s4$			$g8$	$g2$	$g3$
5		$r6$	$r6$		$r6$	$r6$			
6	$s5$			$s4$				$g9$	$g3$
7	$s5$			$s4$					$g10$
8		$s6$			$s11$				
9		$r1$	$s7$		$r1$	$r1$			
10		$r3$	$r3$		$r3$	$r3$			
11		$r5$	$r5$		$r5$	$r5$			

Recognizing Handles

- Given a parse state

Stack	Input
$T*$	$\mathbf{id\$}$

1. Run the DFA on stack, treating shift/goto actions as edges of the DFA: $\mathbf{0} \rightarrow \mathbf{2} \rightarrow \mathbf{7}$.
2. Look up the entry $(\mathbf{7}, \mathbf{id})$ of the transition table: shift 5. (not a handle)
3. Push $\mathbf{id}$ onto the stack.

- Given a parse state

Stack	Input
$T * \mathbf{id}$	$\$$

1. Run the DFA on stack: $\mathbf{0} \rightarrow \mathbf{2} \rightarrow \mathbf{7} \rightarrow \mathbf{5}$.
2. Look up the entry $(\mathbf{5}, \$)$ of the transition table: reduce 6. (handle)
3. Reduce by rule 6: $F \rightarrow \mathbf{id}$

LR Parsing Process

To avoid rescanning the stack for each token, the stack maintains DFA states:

Stack	Symbols	Input	Action
0		$\mathbf{id} * \mathbf{id}\$$	shift to 5
$\mathbf{0\ 5}$	**id**	$*\mathbf{id}\$$	reduce by 6 ($F \rightarrow \mathbf{id}$)
$\mathbf{0\ 3}$	F	$*\mathbf{id}\$$	reduce by 4 ($T \rightarrow F$)
$\mathbf{0\ 2}$	T	$*\mathbf{id}\$$	shift to 7
$\mathbf{0\ 2\ 7}$	$T*$	$\mathbf{id}\$$	shift to 5
$\mathbf{0\ 2\ 7\ 5}$	$T * \mathbf{id}$	$\$$	reduce by 6 ($F \rightarrow \mathbf{id}$)
$\mathbf{0\ 2\ 7\ 10}$	$T * F$	$\$$	reduce by 3 ($T \rightarrow T * F$)
$\mathbf{0\ 2}$	T	$\$$	reduce by 2 ($E \rightarrow T$)
$\mathbf{0\ 1}$	E	$\$$	accept

LR Parsing Algorithm

Repeat the following:

1. Look up top stack state, and input symbol, to get an action.
2. If the action is
    - Shift(n): Advance input one token; push n on stack
    - Reduce(k):
        1. Pop stack as many times as the number of symbols on the right hand side of rule k
        2. Let X be the left-hand-side symbol of rule k
        3. In the state now on top of stack, look up X to get "goto n"
        4. Push n on top of stack
    - Accept: Stop parsing, report success.
    - Error: Stop parsing, report failure.

LR(0) and SLR Parser Generation

For the augmented grammar

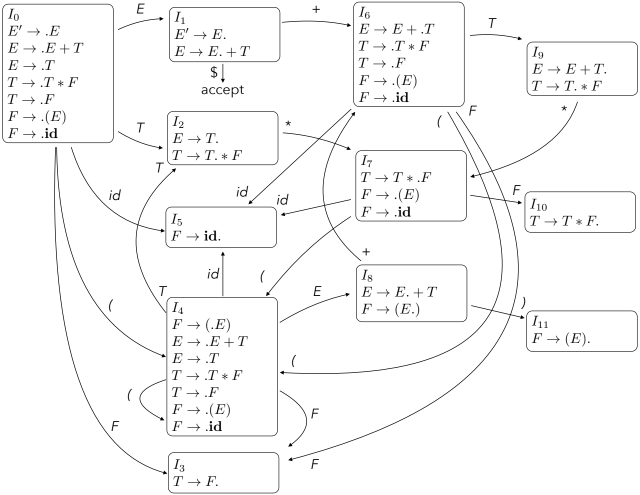
$$\begin{array}{llll} (0) & E' & \rightarrow & E \\ (1) & E & \rightarrow & E + T \\ (2) & E & \rightarrow & T \\ (3) & T & \rightarrow & T * F \\ (4) & T & \rightarrow & F \\ (5) & F & \rightarrow & (E) \\ (6) & F & \rightarrow & \text{id} \end{array}$$

construct the parsing table:

State	id	+	*	(	)	$	E	T	F
0	$s5$			$s4$			$g1$	$g2$	$g3$
1		$s6$				acc			
2		$r2$	$s7$		$r2$	$r2$			
3		$r4$	$r4$		$r4$	$r4$			
4	$s5$			$s4$			$g8$	$g2$	$g3$
5		$r6$	$r6$		$r6$	$r6$			
6	$s5$			$s4$				$g9$	$g3$
7	$s5$			$s4$					$g10$
8		$s6$			$s11$				
9		$r1$	$s7$		$r1$	$r1$			
10		$r3$	$r3$		$r3$	$r3$			
11		$r5$	$r5$		$r5$	$r5$			

LR(0) Automaton

The parsing table is constructed from the LR(0) automaton:



LR(0) Items

A state is a set of *items*.

- An item is a production with a dot somewhere on the body.
- The items for $A \rightarrow XYZ$:

$$\begin{array}{lcl} A & \rightarrow & .XYZ \\ A & \rightarrow & X.YZ \\ A & \rightarrow & XY.Z \\ A & \rightarrow & XYZ. \end{array}$$

- $A \rightarrow \epsilon$ has only one item $A \rightarrow \cdot$.
- An item indicates how much of a production we have seen in parsing.

The Initial Parse State

- Initially, the parser will have an empty stack, and the input will be a complete E-sentence, indicated by item

$$E' \to .E$$

where the dot indicates the current position of the parser.

- Collect all of the items reachable from the initial item without consuming any input tokens:

$$I_0 = \boxed{\begin{array}{ccl} E' & \to & .E \\ E & \to & .E+T \\ E & \to & .T \\ T & \to & .T*F \\ T & \to & .F \\ F & \to & .(E) \\ F & \to & .\mathrm{id} \end{array}}$$

Closure of Item Sets

If I is a set of items for a grammar G, then $CLOSURE(I)$ is the set of items constructed from I by the two rules:

1. Initially, add every item in I to $CLOSURE(I)$.
2. If $A \to \alpha.B\beta$ is in $CLOSURE(I)$ and $B \to \gamma$ is a production, then add the item $B \to .\gamma$ to $CLOSURE(I)$, if it is not already there. Apply this rule until no more new items can be added to $CLOSURE(I)$.

In algorithm:

$CLOSURE(I) =$

repeat

for any item $A \to \alpha.B\beta$ in I

for any production $B \to \gamma$

$I = I \cup \{X \to .\gamma\}$

until I does not change

return I

Construction of LR(0) Automaton

For the initial state

$$I_0 = \boxed{\begin{array}{lcl} E' & \to & .E \\ E & \to & .E + T \\ E & \to & .T \\ T & \to & .T * F \\ T & \to & .F \\ F & \to & .(E) \\ F & \to & .\mathrm{id} \end{array}}$$

construct the next states for each grammar symbol.

Consider E:

1. Find all items of form $A \to \alpha.E\beta$: $\{E' \to .E, E \to .E + T\}$
2. Move the dot over E: $\{E' \to E., E \to E. + T\}$
3. Closure it:

$$I_1 = \boxed{\begin{array}{lcl} E' & \to & E. \\ E & \to & E. + T \end{array}}$$

Construction of LR(0) Automaton

$$I_0 = \boxed{\begin{array}{lcl} E' & \rightarrow & .E \\ E & \rightarrow & .E + T \\ E & \rightarrow & .T \\ T & \rightarrow & .T * F \\ T & \rightarrow & .F \\ F & \rightarrow & .(E) \\ F & \rightarrow & .\mathbf{id} \end{array}}$$

Consider (:

1. Find all items of form $A \rightarrow \alpha.(\beta$: $\{F \rightarrow .(E)\}$
2. Move the dot over E: $\{F \rightarrow (.E)\}$
3. Closure it:

$$I_4 = \boxed{\begin{array}{lcl} F & \rightarrow & (.E) \\ E & \rightarrow & .E + T \\ E & \rightarrow & .T \\ T & \rightarrow & .T * F \\ T & \rightarrow & .F \\ F & \rightarrow & .(E) \\ F & \rightarrow & .\mathbf{id} \end{array}}$$

Exercises

$$I_0 = \boxed{\begin{array}{ccl} E' & \to & .E \\ E & \to & .E+T \\ E & \to & .T \\ T & \to & .T*F \\ T & \to & .F \\ F & \to & .(E) \\ F & \to & .\mathrm{id} \end{array}}$$

- $GOTO(I_0, T) =$
- $GOTO(I_0, F) =$

Goto

When I is a set of items and X is a grammar symbol (terminals and nonterminals, $GOTO(I, X)$ is defined to be the closure of the set of all items $A \to \alpha X.\beta$ such that $A \to \alpha.X\beta$ is in I.
In algorithm:

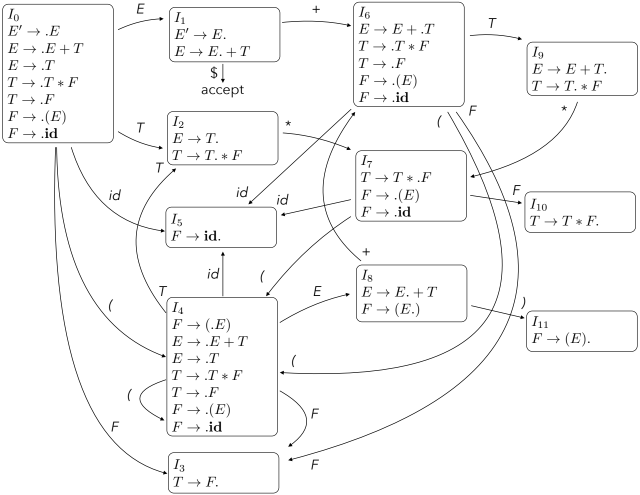
$GOTO(I, X) =$
  set J to the empty set
    for any item $A \to \alpha.X\beta$ in I
      add $A \to \alpha X.\beta$ to J
  return $CLOSURE(J)$

Construction of LR(0) Automaton

- T: the set of states
- E: the set of edges

Initialize T to $\{CLOSURE(\{S' \to S\})\}$
Initialize E to empty
repeat
    for each state I in T
        for each item $A \to \alpha.X\beta$ in I
            let J be $GOTO(I, X)$
            $T = T \cup \{J\}$
            $E = E \cup \{I \xrightarrow{X} J\}$
until E and T do not change

LR(0) Automaton



Construction of LR(0) Parsing Table

- For each edge $I \xrightarrow{X} J$ where X is a terminal, we put the action *shift* J at position (I, X) of the table.
- If X is a nonterminal, we put an *goto* J at position (I, X).
- For each state I containing an item $S' \rightarrow S.$, we put an *accept* action at $(I, \$)$.
- Finally, for a state containing an item $A \rightarrow \gamma.$ (production n with the dot at the end), we put a *reduce* n action at (I, Y) for every token Y.

LR(0) Parsing Table

State	id	$+$	$*$	(	)	\$	E	T	F
0	$s5$			$s4$			$g1$	$g2$	$g3$
1		$s6$				acc			
2	$r2$	$r2$	$r2, s7$	$r2$	$r2$	$r2$			
3	$r4$	$r4$	$r4$	$r4$	$r4$	$r4$			
4	$s5$			$s4$			$g8$	$g2$	$g3$
5	$r6$	$r6$	$r6$	$r6$	$r6$	$r6$			
6	$s5$			$s4$				$g9$	$g3$
7	$s5$			$s4$					$g10$
8		$s6$			$s11$				
9	$r1$	$r1$	$r1, s7$	$r1$	$r1$	$r1$			
10	$r3$	$r3$	$r3$	$r3$	$r3$	$r3$			
11	$r5$	$r5$	$r5$	$r5$	$r5$	$r5$			

Conflicts

The parsing table may contain conflicts (duplicated entries). Two kinds of conflicts:

- Shift/reduce conflicts: the parser cannot tell whether to shift or reduce.
- Reduce/reduce conflicts: the parser knows to reduce, but cannot tell which reduction to perform.

If the LR(0) parsing table for a grammar contains no conflicts, the grammar is in LR(0) grammar.

Construction of SLR Parsing Table

- For each edge $I \xrightarrow{X} J$ where X is a terminal, we put the action *shift* J at position (I, X) of the table.
- If X is a nonterminal, we put an *goto* J at position (I, X).
- For each state I containing an item $S' \rightarrow S.$, we put an *accept* action at $(I, \$)$.
- Finally, for a state containing an item $A \rightarrow \gamma.$ (production n with the dot at the end), we put a *reduce* n action at (I, Y) for every token $Y \in FOLLOW(A)$.

SLR Parsing Table

- In state 2, we have an item $E \to T.$, so we put action $r2$ at the $(2, Y)$ entries where $Y \in FOLLOW(E) = \{\$, +,)\}$
- In state 3, we have an item $T \to F.$, so we put action $r4$ at the $(3, Y)$ entries where $Y \in FOLLOW(T) = \{\$, +,), *\}$
- In state 5, we have an item $F \to \mathbf{id}.$, so we put action $r6$ at the $(5, Y)$ entries where $Y \in FOLLOW(F) = \{\$, +,), *\}$

State	id	+	*	(	)	$	E	T	F
0	$s5$			$s4$			$g1$	$g2$	$g3$
1		$s6$				acc			
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11		$r5$	$r5$		$r5$	$r5$			

LR(0) vs. SLR(1)

- LR(0) parsing makes shift/reduce decisions based solely on the DFA states, which often leads to conflicts (shift-reduce, reduce-reduce).
- SLR addresses this by using a 1-token lookahead. It checks the next token in the current state and uses that information to reduce conflicts. SLR is also called SLR(1).
  - For example, if $A \rightarrow \alpha$ is a candidate for reduction in the current state, the reduction occurs only if the next token is in $FOLLOW(A)$.

Limitations of SLR

Consider the unambiguous grammar:

$$\begin{array}{lcl} S & \rightarrow & L = R \mid R \\ L & \rightarrow & *R \mid \mathbf{id} \\ R & \rightarrow & L \end{array}$$

The LR(0) items include

$$I_2: \quad \begin{array}{l} S \rightarrow L. = R \\ R \rightarrow L. \end{array} \qquad\qquad I_6: \quad \begin{array}{l} S \rightarrow L = .R \\ R \rightarrow .L \\ L \rightarrow . * R \\ L \rightarrow .\mathbf{id} \end{array}$$

where we put the "shift 6" action in the $(2, =)$ entry of the parsing table. Since $=$ is in $FOLLOW(R)$ (since $S \Rightarrow L = R \Rightarrow *R = R$), we also put the "reduce $R \rightarrow L$" action in the entry. E.g.,

Stack	Input	Action
$\$$	$\mathbf{id} = \mathbf{id}\$$	shift
$\$\mathbf{id}$	$= \mathbf{id}\$$	reduce $L \rightarrow \mathbf{id}$
$\$L$	$= \mathbf{id}\$$	shift / reduce $R \rightarrow L$

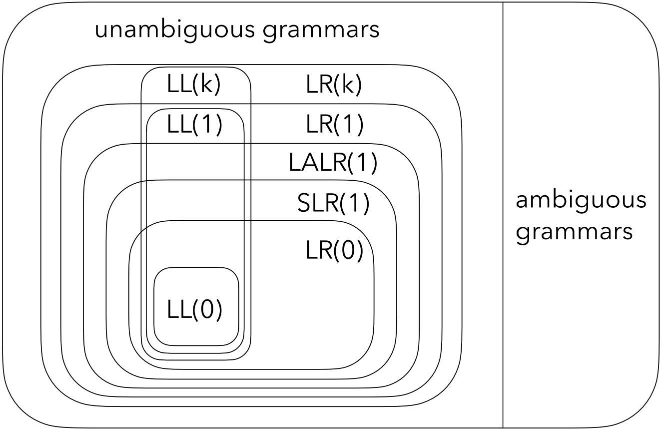
where the correct action is shift in the context of $S \rightarrow L = R$. Reduce $R \rightarrow L$ is only possible when the next token is $\$$. SLR does not consider such a context.

More Powerful LR Parsers

We can extend LR(0) parsing to use one symbol of lookahead on the input:

- LR(1) parsing:
  - The parsing table is based on LR(1) items.
    - E.g., $(R \to L, \$)$: "reduce with $R \to L$ when the next token is $\$$" (do not reduce when the next token is $=$)
  - Make full use of the lookahead symbol.
  - Generate a large set of states.
- LALR(1) parsing.
  - Based on the LR(0) items.
  - Introducing lookaheads into the LR(0) items.
  - Parsing tables have many fewer states than LR(1), no bigger than that of SLR.

Summary


unambiguous grammars
LL(k)
LR(k)
LL(1)
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