

# COSE312: Compilers

## Lecture 14 — Semantic Analysis (4)

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# Denotational Semantics

- In denotational semantics, we are interested in the mathematical meaning of a program.
- *Compositional semantics*: The meaning of an expression is defined with the meanings of its immediate subexpressions.

# Denotational Semantics of Expressions

$a \rightarrow n \mid x \mid a_1 + a_2 \mid a_1 \star a_2 \mid a_1 - a_2$

$b \rightarrow \text{true} \mid \text{false} \mid a_1 = a_2 \mid a_1 \leq a_2 \mid \neg b \mid b_1 \wedge b_2$

$c \rightarrow x := a \mid \text{skip} \mid c_1; c_2 \mid \text{if } b \text{ } c_1 \text{ } c_2 \mid \text{while } b \text{ } c$

$\mathcal{A} \llbracket a \rrbracket : \text{State} \rightarrow \mathbb{Z}$

$\mathcal{A} \llbracket n \rrbracket (s) = n$

$\mathcal{A} \llbracket x \rrbracket (s) = s(x)$

$\mathcal{A} \llbracket a_1 + a_2 \rrbracket (s) = \mathcal{A} \llbracket a_1 \rrbracket (s) + \mathcal{A} \llbracket a_2 \rrbracket (s)$

$\mathcal{A} \llbracket a_1 \star a_2 \rrbracket (s) = \mathcal{A} \llbracket a_1 \rrbracket (s) \times \mathcal{A} \llbracket a_2 \rrbracket (s)$

$\mathcal{A} \llbracket a_1 - a_2 \rrbracket (s) = \mathcal{A} \llbracket a_1 \rrbracket (s) - \mathcal{A} \llbracket a_2 \rrbracket (s)$

$\mathcal{B} \llbracket b \rrbracket : \text{State} \rightarrow \mathbf{T}$

$\mathcal{B} \llbracket \text{true} \rrbracket (s) = \text{true}$

$\mathcal{B} \llbracket \text{false} \rrbracket (s) = \text{false}$

$\mathcal{B} \llbracket a_1 = a_2 \rrbracket (s) = \mathcal{A} \llbracket a_1 \rrbracket (s) = \mathcal{A} \llbracket a_2 \rrbracket (s)$

$\mathcal{B} \llbracket a_1 \leq a_2 \rrbracket (s) = \mathcal{A} \llbracket a_1 \rrbracket (s) \leq \mathcal{A} \llbracket a_2 \rrbracket (s)$

$\mathcal{B} \llbracket \neg b \rrbracket (s) = \mathcal{B} \llbracket b \rrbracket (s) = \text{false}$

$\mathcal{B} \llbracket b_1 \wedge b_2 \rrbracket (s) = \mathcal{B} \llbracket b_1 \rrbracket (s) \wedge \mathcal{B} \llbracket b_2 \rrbracket (s)$

# Denotational Semantics of Commands

$$\mathcal{C} \llbracket x := a \rrbracket (s) = s[x \mapsto \mathcal{A} \llbracket a \rrbracket (s)]$$

$$\mathcal{C} \llbracket \text{skip} \rrbracket = \text{id}$$

$$\mathcal{C} \llbracket c_1; c_2 \rrbracket = \mathcal{C} \llbracket c_2 \rrbracket \circ \mathcal{C} \llbracket c_1 \rrbracket$$

$$\mathcal{C} \llbracket \text{if } b \text{ } c_1 \text{ } c_2 \rrbracket = \text{cond}(\mathcal{B} \llbracket b \rrbracket, \mathcal{C} \llbracket c_1 \rrbracket, \mathcal{C} \llbracket c_2 \rrbracket)$$

$$\mathcal{C} \llbracket \text{while } b \text{ } c \rrbracket = \text{fix } F$$

where

$$\text{cond}(f, g, h) = \lambda s. \begin{cases} g(s) & \dots f(s) = \text{true} \\ h(s) & \dots f(s) = \text{false} \end{cases}$$

$$F(g) = \text{cond}(\mathcal{B} \llbracket b \rrbracket, g \circ \mathcal{C} \llbracket c \rrbracket, \text{id})$$

## Denotational Semantics of Loops

Note that the meaning of the while loop is the mathematical object (i.e. partial function in  $\mathbf{State} \leftrightarrow \mathbf{State}$ ) that satisfies the equation:

$$\mathcal{C}[\text{while } b \text{ } c] = \mathbf{cond}(\mathcal{B}[b], \mathcal{C}[\text{while } b \text{ } c] \circ \mathcal{C}[c], \mathbf{id}).$$

We can rewrite the equation:

$$\mathcal{C}[\text{while } b \text{ } c] = F(\mathcal{C}[\text{while } b \text{ } c])$$

where

$$F(g) = \mathbf{cond}(\mathcal{B}[b], g \circ \mathcal{C}[c], \mathbf{id}).$$

The meaning of the while loop is defined to be the fixed point of  $F$ :

$$\mathcal{C}[\text{while } b \text{ } c] = \mathit{fix}F$$

where  $\mathit{fix}F$  denotes the *least fixed point* of  $F$ .

## Example

`while  $\neg(x = 0)$  skip`

- $F$
- $fix F$

## Some Questions

- Does the least fixed point  $\mathit{fix} F$  exist?
- Is  $\mathit{fix} F$  unique?
- How to compute  $\mathit{fix} F$ ?

# Fixed Point Theory

## Theorem

Let  $f : D \rightarrow D$  be a continuous function on a CPO  $D$ . Then  $f$  has a (unique) least fixed point,  $\mathit{fix}(f)$ , and

$$\mathit{fix}(f) = \bigsqcup_{n \geq 0} f^n(\perp).$$

The denotational semantics is well-defined if

- $\mathbf{State} \hookrightarrow \mathbf{State}$  is a CPO, and
- $F : (\mathbf{State} \hookrightarrow \mathbf{State}) \rightarrow (\mathbf{State} \hookrightarrow \mathbf{State})$  is a continuous function.



# Plan

- Complete Partial Order
- Continuous Functions
- Least Fixed Point

# Partially Ordered Set

## Definition (Partial Order)

We say a binary relation  $\sqsubseteq$  is a partial order on a set  $D$  iff  $\sqsubseteq$  is

- reflexive:  $\forall p \in D. p \sqsubseteq p$
- transitive:  $\forall p, q, r \in D. p \sqsubseteq q \wedge q \sqsubseteq r \implies p \sqsubseteq r$
- anti-symmetric:  $\forall p, q \in D. p \sqsubseteq q \wedge q \sqsubseteq p \implies p = q$

We call such a pair  $(D, \sqsubseteq)$  partially ordered set, or poset.

## Lemma

*If a partially ordered set  $(D, \sqsubseteq)$  has a least element  $d$ , then  $d$  is unique.*

## Exercise

Let  $S$  be a non-empty set. Prove that  $(\mathcal{P}(S), \subseteq)$  is a partially ordered set.

## Exercise

Let  $\mathbf{X} \leftrightarrow \mathbf{Y}$  be the set of all partial functions from a set  $\mathbf{X}$  to a set  $\mathbf{Y}$ , and define  $f \sqsubseteq g$  iff

$$\text{Dom}(f) \subseteq \text{Dom}(g) \wedge \forall x \in \text{Dom}(f). f(x) = g(x).$$

Prove that  $(\mathbf{X} \leftrightarrow \mathbf{Y}, \sqsubseteq)$  is a partially ordered set.

# Least Upper Bound

## Definition (Least Upper Bound)

Let  $(D, \sqsubseteq)$  be a partially ordered set and let  $Y$  be a subset of  $D$ . An upper bound of  $Y$  is an element  $d$  of  $D$  such that

$$\forall d' \in Y. d' \sqsubseteq d.$$

An upper bound  $d$  of  $Y$  is a least upper bound if and only if  $d \sqsubseteq d'$  for every upper bound  $d'$  of  $Y$ . The least upper bound of  $Y$  is denoted by  $\sqcup Y$ .

## Lemma

*If  $Y$  has a least upper bound  $d$ , then  $d$  is unique.*

# Chain

## Definition (Chain)

Let  $(D, \sqsubseteq)$  be a poset and  $Y$  a subset of  $D$ .  $Y$  is called a chain if  $Y$  is totally ordered:

$$\forall y_1, y_2 \in Y. y_1 \sqsubseteq y_2 \text{ or } y_2 \sqsubseteq y_1.$$

## Example

Consider the poset  $(\mathcal{P}(\{a, b, c\}), \subseteq)$ .

- $Y_1 = \{\emptyset, \{a\}, \{a, c\}\}$
- $Y_2 = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$

# Complete Partial Order (CPO)

## Definition (CPO)

A poset  $(D, \sqsubseteq)$  is a CPO, if every chain  $Y \subseteq D$  has  $\bigsqcup Y \in D$ .

## Definition (Complete Lattice)

A poset  $(D, \sqsubseteq)$  is a complete lattice, if every subset  $Y \subseteq D$  has  $\bigsqcup Y \in D$ .

## Lemma

*If  $(D, \sqsubseteq)$  is a CPO, then it has a least element  $\perp$  given by  $\perp = \bigsqcup \emptyset$ .*

## Examples

### Example

Let  $S$  be a non-empty set. Then,  $(\mathcal{P}(S), \subseteq)$  is a CPO. The lub  $\bigsqcup Y$  for  $Y$  is  $\bigcup Y$ . The least element is  $\emptyset$ .



## Examples

### Example

The poset  $(X \leftrightarrow Y, \sqsubseteq)$  of all partial functions from a set  $X$  to a set  $Y$ , equipped with the partial order

$$\text{Dom}(f) \subseteq \text{Dom}(g) \wedge \forall x \in \text{Dom}(f). f(x) = g(x)$$

is a CPO (but not a complete lattice). The lub of a chain  $Y$  is the partial function  $f$  with  $\text{Dom}(f) = \bigcup_{f_i \in Y} \text{Dom}(f_i)$  and

$$f(x) = \begin{cases} f_n(x) & \dots x \in \text{Dom}(f_i) \text{ for some } f_i \in Y \\ \mathbf{undef} & \dots \textit{otherwise} \end{cases}$$

The least element  $\perp = \lambda x. \mathbf{undef}$ .

# Monotone Functions

## Definition (Monotone Functions)

A function  $f : D \rightarrow E$  between posets is *monotone* iff

$$\forall d, d' \in D. d \sqsubseteq d' \implies f(d) \sqsubseteq f(d').$$

## Example

Consider  $(\mathcal{P}(\{a, b, c\}), \subseteq)$  and  $(\mathcal{P}(\{d, e\}), \subseteq)$  and two functions  $f_1, f_2 : \mathcal{P}(\{a, b, c\}) \rightarrow \mathcal{P}(\{d, e\})$

$X$	$\{a, b, c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\emptyset$
$f_1(X)$	$\{d, e\}$	$\{d\}$	$\{d, e\}$	$\{d, e\}$	$\{d\}$	$\{d\}$	$\{e\}$	$\emptyset$

$X$	$\{a, b, c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\emptyset$
$f_2(X)$	$\{d\}$	$\{d\}$	$\{d\}$	$\{e\}$	$\{d\}$	$\{e\}$	$\{e\}$	$\{e\}$

## Exercise

Determine which of the following functionals of

$$(\text{State} \hookrightarrow \text{State}) \rightarrow (\text{State} \hookrightarrow \text{State})$$

are monotone:

①  $F_0(g) = g.$

②  $F_1(g) = \begin{cases} g_1 & \dots g_1 = g_2 \\ g_2 & \dots \textit{otherwise} \end{cases}$  where  $g_1 \neq g_2.$

③  $F_2(g) = \lambda s. \begin{cases} g(s) & \dots s(x) \neq 0 \\ s & \dots s(x) = 0 \end{cases}$

# Properties of Monotone Functions

## Lemma

*Let  $(D_1, \sqsubseteq_1)$ ,  $(D_2, \sqsubseteq_2)$ , and  $(D_3, \sqsubseteq_3)$  be CPOs. Let  $f : D_1 \rightarrow D_2$  and  $g : D_2 \rightarrow D_3$  be monotone functions. Then,  $g \circ f : D_1 \rightarrow D_3$  is a monotone function.*

# Properties of Monotone Functions

## Lemma

Let  $(D_1, \sqsubseteq_1)$  and  $(D_2, \sqsubseteq_2)$  be CPOs. Let  $f : D_1 \rightarrow D_2$  be a monotone function. If  $Y$  is a chain in  $D_1$ , then  $f(Y) = \{f(d) \mid d \in Y\}$  is a chain in  $D_2$ . Furthermore,

$$\bigsqcup f(Y) \sqsubseteq f(\bigsqcup Y).$$

# Continuous Functions

## Definition (Continuous Functions)

A function  $f : D_1 \rightarrow D_2$  defined on posets  $(D_1, \sqsubseteq_1)$  and  $(D_2, \sqsubseteq_2)$  is continuous if it is monotone and it preserves least upper bounds of chains:

$$\bigsqcup f(Y) = f(\bigsqcup Y)$$

for all non-empty chains  $Y$  in  $D_1$ . If  $f(\bigsqcup Y) = \bigsqcup f(Y)$  holds for the empty chain (that is,  $\perp = f(\perp)$ ), then we say that  $f$  is strict.

# Properties of Continuous Functions

## Lemma

*Let  $f : D_1 \rightarrow D_2$  be a monotone function defined on posets  $(D_1, \sqsubseteq_1)$  and  $(D_2, \sqsubseteq_2)$  and  $D_1$  is a finite set. Then,  $f$  is continuous.*



# Properties of Continuous Functions

## Lemma

*Let  $(D_1, \sqsubseteq_1)$ ,  $(D_2, \sqsubseteq_2)$ , and  $(D_3, \sqsubseteq_3)$  be CPOs. Let  $f : D_1 \rightarrow D_2$  and  $g : D_2 \rightarrow D_3$  be continuous functions. Then,  $g \circ f : D_1 \rightarrow D_3$  is a continuous function.*

# Least Fixed Points

## Definition (Fixed Point)

Let  $(D, \sqsubseteq)$  be a poset. A *fixed point* of a function  $f : D \rightarrow D$  is an element  $d \in D$  such that  $f(d) = d$ . We write  $\text{fix}(f)$  for the *least fixed point* of  $f$ , if it exists, such that

- $f(\text{fix}(f)) = \text{fix}(f)$
- $\forall d \in D. f(d) = d \implies \text{fix}(f) \sqsubseteq d$

# Fixed Point Theorem

## Theorem (Kleene Fixed Point)

Let  $f : D \rightarrow D$  be a continuous function on a CPO  $D$ . Then  $f$  has a least fixed point,  $\text{fix}(f)$ , and

$$\text{fix}(f) = \bigsqcup_{n \geq 0} f^n(\perp)$$

$$\text{where } f^n(\perp) = \begin{cases} \perp & n = 0 \\ f(f^{n-1}(\perp)) & n > 0 \end{cases}$$

## Proof

We show the claims of the theorem by showing that  $\bigsqcup_{n \geq 0} f^n(\perp)$  exists and it is indeed equivalent to  $\text{fix}(f)$ . First note that  $\bigsqcup_{n \geq 0} f^n(\perp)$  exists because  $f^0(\perp) \sqsubseteq f^1(\perp) \sqsubseteq f^2(\perp) \sqsubseteq \dots$  is a chain. We show by induction that  $\forall n \in \mathbb{N}. f^n(\perp) \sqsubseteq f^{n+1}(\perp)$ :

- $\perp \sqsubseteq f(\perp)$  ( $\perp$  is the least element)
- $f^n(\perp) \sqsubseteq f^{n+1}(\perp) \implies f^{n+1}(\perp) \sqsubseteq f^{n+2}(\perp)$  (monotonicity of  $f$ )

Now, we show that  $\text{fix}(f) = \bigsqcup_{n \geq 0} f^n(\perp)$  in two steps:

- We show that  $\bigsqcup_{n \geq 0} f^n(\perp)$  is a fixed point of  $f$ :

$$\begin{aligned} f\left(\bigsqcup_{n \geq 0} f^n(\perp)\right) &= \bigsqcup_{n \geq 0} f(f^n(\perp)) && \text{continuity of } f \\ &= \bigsqcup_{n \geq 0} f^{n+1}(\perp) \\ &= \bigsqcup_{n \geq 0} f^n(\perp) \end{aligned}$$

# Proofs

- We show that  $\bigsqcup_{n \geq 0} f^n(\perp)$  is smaller than all the other fixed points. Suppose  $d$  is a fixed point, i.e.,  $f(d) = d$ . Then,

$$\bigsqcup_{n \geq 0} f^n(\perp) \sqsubseteq d$$

since  $\forall n \in \mathbb{N}. f^n(\perp) \sqsubseteq d$ :

$$f^0(\perp) = \perp \sqsubseteq d, \quad f^n(\perp) \sqsubseteq d \implies f^{n+1}(\perp) \sqsubseteq f(d) = d.$$

Therefore, we conclude

$$\text{fix}(f) = \bigsqcup_{n \geq 0} f^n(\perp).$$

# Well-definedness of the Semantics

The function  $F$

$$F(g) = \mathbf{cond}(\mathcal{B}[b], g \circ \mathcal{C}[c], \mathbf{id})$$

is continuous.

## Lemma

Let  $g_0 : \text{State} \hookrightarrow \text{State}$ ,  $p : \text{State} \rightarrow \mathbf{T}$ , and define

$$F(g) = \mathbf{cond}(p, g, g_0).$$

Then,  $F$  is continuous.

## Lemma

Let  $g_0 : \text{State} \hookrightarrow \text{State}$ , and define

$$F(g) = g \circ g_0.$$

Then  $F$  is continuous.