COSE215: Theory of Computation

Lecture 20 — P, NP, and NP-Complete Problems

Hakjoo Oh
2019 Spring

Contents[1]

- $\mathcal{P}$ and $\mathcal{NP}$
- Polynomial-time reductions
- NP-complete problems

[1] Slides are partly based on Siddhartha Sen's ("P, NP, and NP-Completeness")

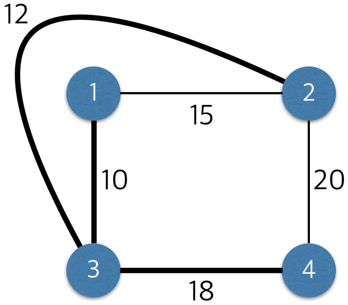
Problems Solvable in Polynomial Time ($\mathcal{P}$)

- A Turing machine M is said to be of time complexity $T(n)$ if whenever M is given an input w of length n, M halts after making at most $T(n)$ moves, regardless of whether or not M accepts.
  - E.g., $T(n) = 5n^2$, $T(n) = 3^n + 5n^4$
- Polynomial time: $T(n) = a_0 n^k + a_1 n^{k-1} + \cdots + a_k n + a_{k+1}$
- We say a language L is in class $\mathcal{P}$ if there is some polynomial $T(n)$ such that $L = L(M)$ for some deterministic TM M of time complexity $T(n)$.
- Problems solvable in polynomial time are called *tractable*.

Example: Kruskal's Algorithm

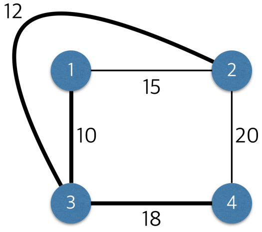
A greedy algorithm for finding a minimum-weight spanning tree for a weighted graph.

- a spanning tree: a subset of the edges such that all nodes are connected through these edges
- a minimum-weight spanning tree: a spanning tree with the least total weight



Example: Kruskal's Algorithm

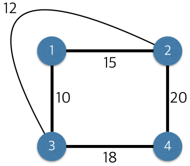
- Consider the edge (1,3) with the lowest weight (10). Because nodes 1 and 3 are not contained in $\boldsymbol{T}$ at the same time, include the edge in $\boldsymbol{T}$.
- Consider the next edge in order of weights: (2,3). Since 2 and 3 are not in $\boldsymbol{T}$ at the same time, include (2,3) in $\boldsymbol{T}$.
- Consider the next edge: (1,2). Nodes 1 and 2 are in $\boldsymbol{T}$. Reject (1,2).
- Consider the next edge (3,4) and include it in $\boldsymbol{T}$.
- We have three edges for the spanning tree of a 4-node graph, so stop.



The algorithm takes $O(m + e \log e)$ steps ($O(n^2)$ for multitape TM).

Nondeterministic Polynomial Time ($\mathcal{NP}$)

- We say a language L is in the class $\mathcal{NP}$ (nondeterministic polynomial) if there is a nondeterministic TM M and a polynomial time complexity $T(n)$ such that $L = L(M)$, and when M is given an input of length n, there are no sequences of more than $T(n)$ moves of M.
- Example: TSP (Travelling Salesman Problem)
  - finding a *hamiltonian cycle* (i.e., a cycle that contains all nodes and each node exactly once) with minimum cost: e.g.,



  - To solve TSP, we need to try an exponential number of cycles and compute their total weight. Thus, TSP may not be in $\mathcal{P}$. TSP is in $\mathcal{NP}$ because NTM can guess an exponential number of possible solutions and checking a hamiltonian cycle can be done in polynomial time.

$\mathcal{P} = \mathcal{NP}$?

One of the deepest open problems.

- In words: everything that can be done in polynomial time by an NTM can in fact be done by a DTM in polynomial time?
- $\mathcal{P} \subseteq \mathcal{NP}$ because every deterministic TM is a nondeterministic TM.
- $\mathcal{P} \supseteq \mathcal{NP}$? Probably not. It appears that $\mathcal{NP}$ contains many problems not in $\mathcal{P}$. However, no one proved it.

Implications of $\mathcal{P} = \mathcal{NP}$

If P=NP, then the world would be a profoundly different place than we usually assume it to be. There would be no special value in "creative leaps," no fundamental gap between solving a problem and recognizing the solution once it's found. Everyone who could appreciate a symphony would be Mozart; everyone who could follow a step-by-step argument would be Gauss; everyone who could recognize a good investment strategy would be Warren Buffett.

— Scott Aaronson

NP-Complete Problems

- NP-complete problems are the "hardest" problems in the NP class.
- If any NP-complete problem can be solved in polynomial time, then all problems in NP are solvable in polynomial time.
- How to compare easiness/hardness of problems?

Problem Solving by Reduction

- L_1: the language (problem) to solve
- L_2: the problem for which we have an algorithm to solve
- Solve L_1 by reducing L_1 to L_2 ($L_1 \leq L_2$) via function f:
  1. Convert input x of L_1 to instance $f(x)$ of L_2
     - ★ $x \in L_1 \iff f(x) \in L_2$
  2. Apply the algorithm for L_2 to $f(x)$
- Running time = time to compute f + time to apply algorithm for L_2
- We write $L_1 \leq_P L_2$ if $f(x)$ is computable in polynomial time

Reductions show easiness/hardness

- To show L_1 is easy, reduce it to something we know is easy
  - $L_1 \leq_P easy$
  - Use algorithm for easy language to decide L_1
- To show L_1 is hard, reduce something we know is hard to it (e.g., NP-complete problem)
  - $hard \leq_P L_1$
  - If L_1 was easy, *hard* would be easy too

NP-Complete Problems

We say L is NP-complete if

1. L is in $\mathcal{NP}$
2. For every language L' in $\mathcal{NP}$, there is a polynomial time reduction of L' to L (i.e., $L' \leq_P L$)

The Boolean Satisfiability Problem

Determine if the given boolean formula can be true.

- $x \land \neg x$
- $x \land \neg(y \lor z)$

The *first* problem proven to be NP-complete.

Theorem (Cook-Levin)

SAT is NP-complete.

We need to show that

1. SAT is NP, and
2. for *every* L in NP, there is a polynomial-time reduction of L to SAT.

Many problems in artificial intelligence, automatic theorem proving, circuit design, etc reduce to the SAT problem.

Summary

The classes of problems that we have considered:

- Undecidable
  - Recursively enumerable
  - Not recursively enumerable
- Decidable
  - $\mathcal{P}$
  - $\mathcal{NP}$
  - NP-complete