COSE215: Theory of Computation

Lecture 19 — Undecidability

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Recursively Enumerable Languages

Definition

A language L is *recursively enumerable* (RE) if there exists a Turing machine that accepts it.

That is, we call a language recursively enumerable if $L = L(M)$ for some Turing machine M:

$$L \text{ is RE} \iff \exists M \in TM.\ L = L(M).$$

In other words, if L is recursively enumerable, there exists a Turing machine M such that

$$\forall w \in L.\ q_0 w \vdash^* x_1 q_f x_2 \text{ and } \forall w \notin L.\ q_0 w \not\vdash^* x_1 q_f x_2$$

Note that when $w \notin L$, the machine may either halt in a non-final state or goes into an infinite loop and run forever.

Recursive Languages (Decidable Languages)

Definition

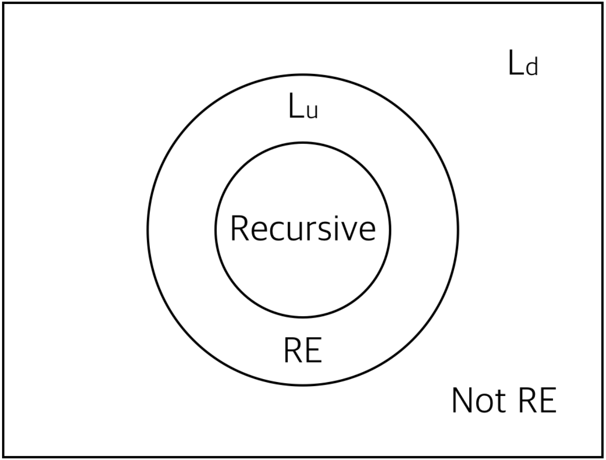
A language L is *recursive* if there exists a Turing machine that accepts it and always terminates.

In other words, we call a language L *recursive* if $L = L(M)$ for some Turing machine M such that:

1. If w is in L, then M accepts (and therefore halts)
2. If w is not in L, then M eventually halts (in a non-final state)

A Turing machine of this type corresponds to our notion of "algorithm", a well-defined sequence of steps that always terminates and produces an answer. When we think of the language L as a "problem", L is called *decidable* if it is recursive, otherwise called *undecidable*.

Overview



L_d: A language that is not recursively enumerable

L_d is not recursively enumerable and therefore we cannot design a Turing machine to accept L_d. The definition of L_d:

$$L_d = \{w_i \mid w_i \notin L(M_i)\}$$

To understand the definition, we need two preliminary concepts:

1. Enumerating binary strings
2. Representing Turing machines in binary strings

Enumerating Binary Strings

- Every binary string can be represented by a unique integer i:

  The integer for binary string w is the integer value of $1w$.

  For instance, ϵ has 1, 0 has 2, 1 has 3, 00 has 4, and so on.
- Using this representation, we can order binary strings; ϵ is the first string, 0 is the second, 1 the third, 00 the fourth, and so on. Let w_i the ith binary string:

$$
\begin{array}{rcl}
w_1 & = & \epsilon \\
w_2 & = & 0 \\
w_3 & = & 1 \\
w_4 & = & 00 \\
w_5 & = & 01 \\
w_6 & = & 10 \\
w_7 & = & 11 \\
w_8 & = & 000 \\
w_9 & = & 001 \\
 & \vdots &
\end{array}
$$

Representing Turing Machines as Binary Strings

$$M = (Q, \{0, 1\}, \Gamma, \delta, q_1, B, F)$$

1. $Q = \{q_1, q_2, \ldots, q_r\}$
2. $\Gamma = \{X_1, X_2, X_3, \ldots, X_s\}$
3. Directions : $\{D_1, D_2\}$

Encoding for the transition function $\delta(q_i, X_j) = (q_k, X_l, D_m)$:

$$0^i10^j10^k10^l10^m$$

Encoding for the Turing machine M:

$$C_1 11 C_2 11 \cdots C_{n-1} 11 C_n$$

(C_i: encoding for the ith transition rule of M).

Example

$$M = (\{q_1, q_2, q_3\}, \{0, 1\}, \{0, 1, B\}, \delta, q_1, B, \{q_2\})$$

$$\begin{array}{rclr} \delta(q_1, 1) & = & (q_3, 0, R), & 0100100010100 \\ \delta(q_3, 0) & = & (q_1, 1, R), & 0001010100100 \\ \delta(q_3, 1) & = & (q_2, 0, R), & 00010010010100 \\ \delta(q_3, B) & = & (q_3, 1, L), & 0001000100010010 \end{array}$$

Encoding in binary string:

$$01001000101001100010101001001100010010010100110001000100010010$$

Turing machines can be ordered

M_i: The ith Turing machine

Definition

We define M_i to be the Turing machine whose binary representation is w_i.

Many integers do not correspond to any TM at all. For instance, there is not Turing machine whose binary representation is $w_4 = 00$. When M_i is not a valid Turing machine, define M_i to be a Turing machine with one state and no transitions, e.g., M_4. In this case, the machine halts immediately on any input. Thus, $L(M_i) = \emptyset$ if w_i is not a valid TM code.

The Diagonalization Language

Definition

$L_d = \{w_i \mid w_i \notin L(M_i)\}$

That is, L_d is the set of all strings w such that TM M whose binary code is w does not accept w.

Theorem

L_d is not a recursively enumerable language.

Proof Sketch

Consider the table T:

		j			
		1	**2**	**3**	**4**	$\cdots$
	1	0	1	1	0	$\cdots$
i	**2**	1	1	0	0	$\cdots$
	3	0	0	1	1	$\cdots$
	4	0	1	0	1	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

- $T[i, j]$ is 1 iff TM M_i accepts input string w_j. The ith row is the characteristic vector for the language $L(M_i)$.
- To construct L_d, complment the diagonal.
- No Turing machine exists whose language is the diagonal language L_d, because the characteristic vector for L_d (i.e., the complemented diagonal) is different from the characteristic vectors of all Turing macines.
- The language of M_1 is not L_d because $L(M_1)$ differs from L_d for w_1, the language of M_2 differes from L_d for w_2, and so on.

L_u: A language that is RE but not recursive

The following language is recursively enumerable but not recursive:

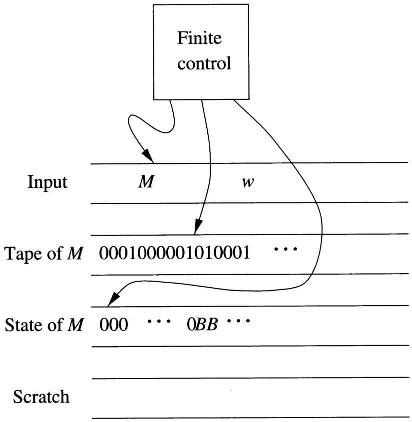
$$L_u = \{(M, w) \mid w \in L(M)\}$$

where M denotes Turing machines in binary strings. L_u is the set of strings representing a TM and an input accepted by the TM.

We can show that there is a Turing machine U, called *universal Turing machine*, such that $L_u = L(U)$. Given a Turing machine M and its input w, U executes M on w and checks whether the machine M accepts the string w or not.

Universal Turing Machine

A multitape Turing machine:



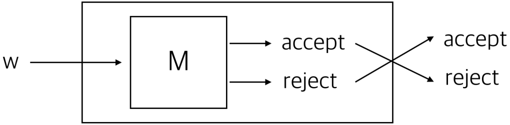
Universal Turing Machine

- In the first tape, the binary representation of M (i.e. transitions of M) and the input string w are stored.
- The second tape will be used to hold the simulated tape of M, where the tape symbols are represented by binary strings. That is, tape symbol X_i of M will be represented by $\mathbf{0}^i$, and tape symbols are separated by single $\mathbf{1}$'s.
- The third tape of U holds the state of M, with state q_i represented by i 0's.
- To simulate a move of M, U searches on its first tape for a transition $\mathbf{0}^i\mathbf{1}\mathbf{0}^j\mathbf{1}\mathbf{0}^k\mathbf{1}\mathbf{0}^l\mathbf{1}\mathbf{0}^m$, such that $\mathbf{0}^i$ is the state on tape $\mathbf{3}$, and $\mathbf{0}^j$ is the tape symbol of M that begins at the position on tape $\mathbf{2}$ scanned by U. This transition is the one M would next make. U should:
  1. Change the contents of tape 3 to $\mathbf{0}^k$.
  2. Replace $\mathbf{0}^j$ on tape 2 by $\mathbf{0}^l$.
  3. Move the head on tape 2 to the position of the next 1 to the left or right, respectively, depending on whether $m = 1$ (move left) or $m = 2$ (move right).
- U accepts the coded pair (M, w) if and only if M accepts w.

Properties of complements (1)

Lemma

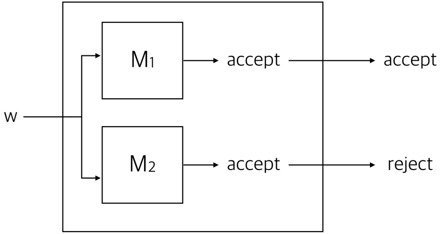
If L is a recursive language, then so is $\bar{L}$.



Properties of Complements (2)

Lemma

If both a language L and its complement are RE, then L is recursive.



L_u is not recursive

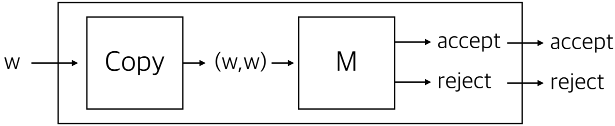
Theorem

L_u is RE but not recursive.

- Suppose L_u were recursive.
- Then by the property of complements, $\bar{L_u}$ is also recursive.
- However, if we have a TM M to accept $\bar{L_u}$, then we can construct a TM to accept L_d (explained next).
- We already know that L_d is not RE, contradiction.

Construction of TM to accept L_d from TM to accept $\bar{L_u}$

Suppose $L(M) = \bar{L_u}$. We construct M' s.t. $L(M') = L_d$ as follows:



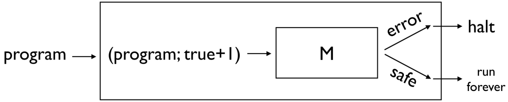
Summary

We have

1. defined the class of recursively enumerable languages,
2. defined the class of recursive languages,
3. defined a non-recursively enumerable language L_d, and
4. defined a non-recursive language L_u.

cf) Undecidable Problems in Practice

- Halting problem
  - See https://www.youtube.com/watch?v=92WHN-pAFCs
- Reasoning about programs



- Properties of CFG:
  - Is a given CFG ambiguous?
  - For CFGs G_1 and G_2, is $L(G_1) \cap L(G_2) = \emptyset$?
  - Is $L(G_1) = L(G_2)$?
  - Is $L(G_1) = L(R)$ for some regular expression R?
  - Is $L(G_1) = T^*$ for some alphabet T?
- Post's Correspondence Problem (PCP)

cf) Post's Correspondence Problem

We can describe this problem as a type of puzzle. We are given a collection of dominos, each containing two strings, one on each side: e.g.,

$$\{[\frac{b}{ca}], [\frac{a}{ab}], [\frac{ca}{a}], [\frac{abc}{c}]\}$$

The task is to make a list of dominos (repetitions permitted) so that the string we get by reading the symbols on the top is the same as the string of symbols on the bottom. This list is called a *match*: e.g.,

$$[\frac{a}{ab}][\frac{b}{ca}][\frac{ca}{a}][\frac{a}{ab}][\frac{abc}{c}]$$

For some collection of dominos, finding a match may not be possible: e.g.,

$$\{[\frac{abc}{ab}], [\frac{ca}{a}], [\frac{acc}{ba}]\}$$

Post's Correspondence Problem is to determine whether a collection of dominos has a match.

Announcement: Final Exam

- We do have class on June 17 (Mon)
- Final exam will be on June 19 (Wed), in class
  - Coverage: Lectures 10–20