# COSE215: Theory of Computation

# Lecture 4 — Nondeterministic Finite Automata

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# Definition

## Definition (NFA)

A *nondeterministic finite automaton* (or *NFA*) is defined as,

$$M = (Q, \Sigma, \delta, q_0, F)$$

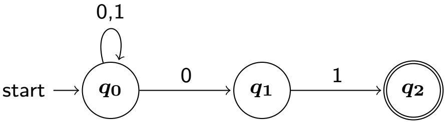
where

- $Q$: a finite set of *states*
- $\Sigma$: a finite set of *input symbols* (or input alphabet)
- $q_0 \in Q$: the *initial state*
- $F \subseteq Q$: a set of *final states*
- $\delta : Q \times \Sigma \to 2^Q$: *transition function*

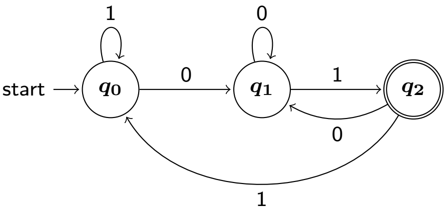
# Example

$$(\{q_0, q_1, q_2\}, \{0, 1\}, \delta, q_0, \{q_2\})$$

$$\begin{array}{ll} \delta(q_0, 0) = \{q_0, q_1\} & \delta(q_0, 1) = \{q_0\} \\ \delta(q_1, 0) = \emptyset & \delta(q_1, 1) = \{q_2\} \\ \delta(q_2, 0) = \emptyset & \delta(q_2, 1) = \emptyset \end{array}$$




cf) Compare with the equivalent DFA:

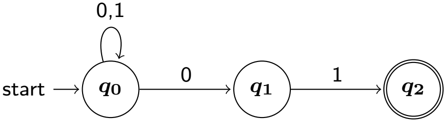


# Extended Transition Function

$$\delta^* : Q \times \Sigma^* \rightarrow 2^Q$$

- (Basis) $s = \epsilon$:
- (Induction) $s = wa$:

# Example




$$\delta^*(q_0, 00101) = \bigcup_{s_i \in \delta^*(q_0, 0010)} \delta(s_i, 1) = \delta(q_0, 1) \cup \delta(q_1, 1) = \{q_0\} \cup \{q_2\} = \{q_0, q_2\}$$

$$\delta^*(q_0, 0010) = \bigcup_{s_i \in \delta^*(q_0, 001)} \delta(s_i, 0) = \delta(q_0, 0) \cup \delta(q_2, 0) = \{q_0, q_1\} \cup \emptyset = \{q_0, q_1\}$$

$$\delta^*(q_0, 001) = \bigcup_{s_i \in \delta^*(q_0, 00)} \delta(s_i, 1) = \delta(q_0, 1) \cup \delta(q_1, 1) = \{q_0\} \cup \{q_2\} = \{q_0, q_2\}$$

$$\delta^*(q_0, 00) = \bigcup_{s_i \in \delta^*(q_0, 0)} \delta(s_i, 0) = \delta(q_0, 0) \cup \delta(q_1, 0) = \{q_0, q_1\} \cup \emptyset = \{q_0, q_1\}$$

$$\delta^*(q_0, 0) = \bigcup_{s_i \in \delta^*(q_0, \epsilon)} \delta(s_i, 0) = \delta(q_0, 0) = \{q_0, q_1\}$$

$$\delta^*(q_0, \epsilon) = \{q_0\}$$

# Exercise: Language of an NFA

The language of NFA $M = (Q, \Sigma, \delta, q_0, F)$ is defined as follows:

$$L(M) = \{ \qquad \qquad \qquad \qquad \qquad \qquad \}$$

# Exercises

Design NFAs for the following languages:

1. $L = \{a^n b \mid n \geq 0\}$
2. $L = \{x01y \mid x, y \in \{0,1\}^*\}$
3. $L = \{01w \mid w \in \{0,1\}^*\}$
4. $L = \{w \in \{0,1\}^* \mid w$ contains at least two $0$'s$\}$
5. $L = \{w \in \{0,1\}^* \mid w$ contains exactly two $0$'s$\}$
6. $L = \{w \in \{0,1\}^* \mid w$ has three consecutive $0$'s$\}$

# Equivalence of DFA and NFA

**Theorem (Equivalence)**

*A Language $L$ is accepted by some NFA if and only if $L$ is accepted by some DFA.*

**Proof.**

By the two Lemmas below. □

**Lemma (DFA to NFA)**

*Given a DFA $D$, there always exists an NFA $N$ such that $L(D) = L(N)$.*

**Lemma (NFA to DFA)**

*Given an NFA $N$, there always exists a DFA $D$ such that $L(N) = L(D)$.*

# DFA to NFA

## Lemma (DFA to NFA)

*Given a DFA $D$, there always exists an NFA $N$ such that $L(D) = L(N)$.*

Proof) Assume a DFA $D = (Q, \Sigma, \delta_D, q_0, F)$ is given. Define an NFA as follows:

$$N = (Q, \Sigma, \delta_N, q_0, F) \text{ where } \delta_N(q, a) = \{\delta_D(q, a)\}$$

To prove:

$$L(D) = \{w \in \Sigma^* \mid \delta_D^*(q_0, w) \in F\} = \{w \in \Sigma^* \mid \delta_N^*(q_0, w) \cap F \neq \emptyset\} = L(N)$$

It is enough to show that

$$\delta_N^*(q_0, w) = \{\delta_D^*(q_0, w)\}$$

The proof is by induction on $|w|$.

- $w = \epsilon$: By the definitions of $\delta_D^*$ and $\delta_N^*$, $\delta_D^*(q_0, \epsilon) = q_0$ and $\delta_N^*(q_0, \epsilon) = \{q_0\}$.
- $w = sa$:

$$\begin{aligned}
\delta_N^*(q_0, sa) &= \bigcup_{s_i \in \delta_N^*(q_0, s)} \delta_N(s_i, a) && \text{by definition of } \delta_N^* \\
&= \delta_N(\delta_D^*(q_0, s), a) && \text{by I.H.} \\
&= \{\delta_D(\delta_D^*(q_0, s), a)\} && \text{by definition of } \delta_N \\
&= \{\delta_D^*(q_0, sa)\} && \text{by definition of } \delta_D^*
\end{aligned}$$

# NFA to DFA (Subset Construction)

## Lemma (NFA to DFA)

*Given an NFA $N$, there always exists a DFA $D$ such that $L(N) = L(D)$.*

Proof) Assume an NFA $N = (Q_N, \Sigma, \delta_N, q_0, F_N)$. Define a DFA as follows

$$D = (Q_D, \Sigma, \delta_D, \{q_0\}, F_D)$$

where

- $Q_D = 2^{Q_N}$
- $F_D = \{S \in Q_D \mid S \cap F_N \neq \emptyset\}$.
- For each $S \in Q_D$ and input symbol $a \in \Sigma$:

$$\delta_D(S, a) = \bigcup_{p \in S} \delta_N(p, a)$$

# NFA to DFA

Then, we can prove $L(N) = L(D)$ by showing that

$$\delta_D^*(\{q_0\}, w) = \delta_N^*(q_0, w).$$

The proof is by induction on the length of $w$.

- $w = \epsilon$: By definition, $\delta_D^*(\{q_0\}, \epsilon) = \{q_0\} = \delta_N^*(q_0, \epsilon)$.
- $w = sa$: Induction hypothesis (I.H.):

$$\delta_D^*(\{q_0\}, s) = \delta_N^*(q_0, s).$$

$$\begin{aligned}
\delta_D^*(\{q_0\}, sa) &= \delta_D(\delta_D^*(\{q_0\}, s), a) && \text{by definition of } \delta_D^* \\
&= \delta_D(\delta_N^*(q_0, s), a) && \text{by I.H.} \\
&= \bigcup_{p \in \delta_N^*(q_0, s)} \delta_N(p, a) && \text{by definition of } \delta_D \\
&= \delta_N^*(q_0, sa) && \text{by definition of } \delta_N^*
\end{aligned}$$

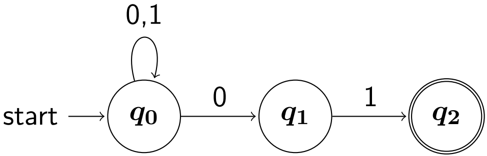
□

# Example: Subset Construction

Find a DFA that is equivalent to:

$$N = (\{q_0, q_1, q_2\}, \{0, 1\}, \delta, q_0, \{q_2\})$$

$$\begin{array}{ll} \delta(q_0, 0) = \{q_0, q_1\} & \delta(q_0, 1) = \{q_0\} \\ \delta(q_1, 0) = \emptyset & \delta(q_1, 1) = \{q_2\} \\ \delta(q_2, 0) = \emptyset & \delta(q_2, 1) = \emptyset \end{array}$$



# Example: Subset Construction

$$D = (Q_D, \{0,1\}, \delta_d, \{q_0\}, F_D)$$

- $Q_D = 2^{\{q_0,q_1,q_2\}} = \{\emptyset, \{q_0\}, \{q_1\}, \ldots, \{q_0, q_1, q_2\}\}$
- $F_D = \{\{q_2\}, \{q_0, q_2\}, \{q_1, q_2\}, \{q_0, q_1, q_2\}\}$
- $\delta_D$:

| | 0 | 1 |
|---|---|---|
| $\emptyset$ | $\emptyset$ | $\emptyset$ |
| $\rightarrow \{q_0\}$ | $\{q_0, q_1\}$ | $\{q_0\}$ |
| $\{q_1\}$ | $\emptyset$ | $\{q_2\}$ |
| $*\{q_2\}$ | $\emptyset$ | $\emptyset$ |
| $\{q_0, q_1\}$ | $\{q_0, q_1\}$ | $\{q_0, q_2\}$ |
| $*\{q_0, q_2\}$ | $\{q_0, q_1\}$ | $\{q_0\}$ |
| $*\{q_1, q_2\}$ | $\emptyset$ | $\{q_2\}$ |
| $*\{q_0, q_1, q_2\}$ | $\{q_0, q_1\}$ | $\{q_0, q_2\}$ |

# Example: Subset Construction