COSE215: Theory of Computation

Lecture 1 — Mathematical Preliminaries

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Today

- Icebreaking: Introduce yourself
- Mathematical backgrounds and notation
  - Sets
  - Inductive proofs

Icebreaking

Introduce yourself:

- Free format. Say anything.
- Nothing to talk about? major, grade, interests, hobbies, specialty, goal, motivation for this course, what you expect from this course, etc

Sets

- A set is a collection of elements, e.g.,
  - $S = \{0, 1, 2\} = \{x \in \mathbb{N} \mid 0 \leq x \leq 2\}$
  - $S = \{2, 4, 6, \ldots\} = \{x \in \mathbb{N} \mid x \text{ is even}\}$
- Notations:
  - $\emptyset$: the empty set
  - $S_1 \subseteq S_2$ iff $\forall x \in S_1.\ x \in S_2$
  - $S_1 \subset S_2$ if $S_1 \subseteq S_2$ and $S_1 \neq S_2$, e.g., $\{1, 2\} \subset \{1, 2, 3\}$, $\{1, 2\} \not\subset \{1, 2\}$
  - $|S|$: the number of elements in set S
  - S_1 and S_2 are disjoint iff $S_1 \cap S_2 = \emptyset$.

Construction of Sets

- Union, intersection, and difference:

$$
\begin{aligned}
S_1 \cup S_2 &= \{x \mid x \in S_1 \vee x \in S_2\} \\
S_1 \cap S_2 &= \{x \mid x \in S_1 \wedge x \in S_2\} \\
S_1 - S_2 &= \{x \mid x \in S_1 \wedge x \notin S_2\}
\end{aligned}
$$

- $\overline{S} = \{x \mid x \in U \wedge x \notin S\}$
- Powerset: $2^S = \mathcal{P}(S) = \{x \mid x \subseteq S\}$
- Cartesian product:

$$S_1 \times S_2 = \{(x, y) \mid x \in S_1 \wedge y \in S_2\}$$

In general,

$$S_1 \times S_2 \times \cdots \times S_n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in S_i\}$$

Partition

When $S_1, S_2, \ldots, S_n$ are subsets of a given set S, $S_1, S_2, \ldots, S_n$ forms a partition of S iff:

1. $S_1, S_2, \ldots, S_n$ are mutually disjoint:

$$\forall i, j.\ i \neq j \implies S_i \cap S_j = \emptyset$$

2. $S_1, S_2, \ldots, S_n$ cover S:

$$\bigcup_{1 \leq i \leq n} S_i = S$$

3. none of S_i is empty: $\forall i. S_i \neq \emptyset$.

Inductive proofs

In CS, every set is inductively defined. E.g.,

Example (Inductive Definition of Trees)

A set of trees is defined as follows:

1. (Basis) A single node (called root) is a tree.
2. (Induction) If $T_1, T_2, \ldots, T_k$ are trees, then the following is also a tree:
   1. Begin with a new node N, which is the root of the tree.
   2. Add edges from N to the roots of each of the trees $T_1, T_2, \ldots, T_k$.

Example (Inductive Definition of Arithmetic Expressions)

A set of arithmetic expressions is defined as follows:

- (Basis) Any number or letter (i.e., a variable) is an expression.
- (Induction) If E and F are expressions, then so are $E + F$, $E * F$, and (E).

Inductive Proofs

Induction is used to prove properties about inductively defined sets. Let S be an inductively-defined set. Let $P(x)$ be a property of x. To show that, for all $x \in S.P(x)$, it suffices to show that:

1. (Base case): Show $P(x)$ for all basis elements $x \in S$.
2. (Inductive case): For each inductive rule using elements $x_1, \ldots, x_k$ of S to construct an element x, show that

$$\text{if } P(x_1), \ldots, P(x_k) \text{ then } P(x)$$

$P(x_1), \ldots, P(x_k)$: induction hypotheses.

Inductive Proofs: Example

Prove that every tree has one more node than it has edges.

Proof.

Formally, what we prove is $P(T) =$ "if T is a tree, and T has n nodes and e edges, then $n = e + 1$".

1. Base case: The base case is when T is a single node. Then, $n = 1$ and $e = 0$, so the relationship $n = e + 1$ holds.
2. Inductive case: The inductive case is when T is built with root node N and k smaller trees $T_1, T_2, \ldots, T_k$.
   1. **Induction hypothesis:** The statements $P(T_i)$ holds for $i = 1, 2, \ldots, k$. That is T_i have n_i nodes and e_i edges; then $n_i = e_i + 1$.
   2. **To Show:** $P(T)$ holds: if T has n nodes and e edges, then $n = e + 1$. The nodes of T are node N and all the nodes of the T_i's, i.e., $n = 1 + n_1 + \cdots + n_k$ The edges of T are the k edges we added explicitly in the inductive definition step, plus the edges of the T_i's. Hence, T has $e = k + e_1 + \cdots + e_k$ edges.

$$
\begin{aligned}
n &= 1 + n_1 + \cdots + n_k && \text{def. of } n \\
&= 1 + (e_1 + 1) + \cdots + (e_k + 1) && \text{induction hypothesis} \\
&= 1 + k + e_1 + \cdots + e_k && \\
&= 1 + e && \text{def. of } e
\end{aligned}
$$

□

Inductive Proofs: Example

Prove that every expression has an equal number of left and right parentheses.

Proof.

Formally, the formal statement $P(G)$ we need to prove is: "if G has l left parentheses and r right parentheses, then $l = r$."

1. **Base case:** The base case is when G is a number or a variable, in which cases $l = r = 0$.
2. **Inductive case:** There are three cases, where G is constructed recursively from smaller expressions:
   - $G = E + F$:
     1. **Induction hypothesis:** The statement holds for all smaller expressions: for E, $l_E = r_E$, and for F, $l_F = r_F$.
     2. **To Show**: $P(G)$ holds: $l_G = r_G$:

$$\begin{aligned} l_G &= l_E + l_F \\ &= r_E + r_F \quad \text{I.H.} \\ &= r_G \end{aligned}$$

   - $G = E * F$: similar
   - $G = (E)$: similar

□

Summary

- Sets: definition, notations, constructions
- Inductive definitions and proofs.