COSE215: Theory of Computation

Lecture 3-2 — Nondeterministic Finite Automata

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Definition

Definition (NFA)

A *nondeterministic finite automaton* (or *NFA*) is defined as,

$$M = (Q, \Sigma, \delta, q_0, F)$$

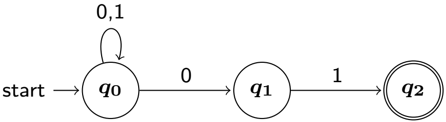
where

- Q: a finite set of *states*
- Σ: a finite set of *input symbols* (or input alphabet)
- $q_0 \in Q$: the *initial state*
- $F \subseteq Q$: a set of *final states*
- $\delta : Q \times \Sigma \to 2^Q$: *transition function*

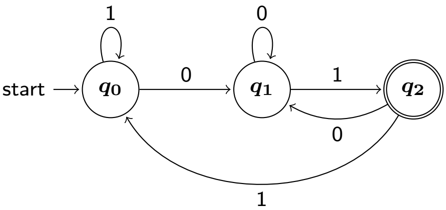
Example

$$(\{q_0, q_1, q_2\}, \{0, 1\}, \delta, q_0, \{q_2\})$$

$$\begin{array}{ll} \delta(q_0, 0) = \{q_0, q_1\} & \delta(q_0, 1) = \{q_0\} \\ \delta(q_1, 0) = \emptyset & \delta(q_1, 1) = \{q_2\} \\ \delta(q_2, 0) = \emptyset & \delta(q_2, 1) = \emptyset \end{array}$$



cf) Compare with the equivalent DFA:



Extended Transition Function

$$\delta^* : Q \times \Sigma^* \to 2^Q$$

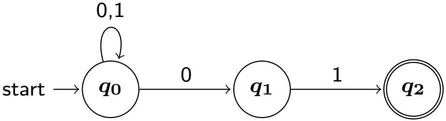
- (Basis) $s = \epsilon$:

$$\delta^*(q, \epsilon) = \{q\}$$

- (Induction) $s = wa$:

$$\delta^*(q, wa) = \bigcup_{s_i \in \delta^*(q,w)} \delta(s_i, a)$$

Example



$$\delta^*(q_0, 00101) = \bigcup_{s_i \in \delta^*(q_0, 0010)} \delta(s_i, 1) = \delta(q_0, 1) \cup \delta(q_1, 1) = \{q_0\} \cup \{q_2\} = \{q_0, q_2\}$$

$$\delta^*(q_0, 0010) = \bigcup_{s_i \in \delta^*(q_0, 001)} \delta(s_i, 0) = \delta(q_0, 0) \cup \delta(q_2, 0) = \{q_0, q_1\} \cup \emptyset = \{q_0, q_1\}$$

$$\delta^*(q_0, 001) = \bigcup_{s_i \in \delta^*(q_0, 00)} \delta(s_i, 1) = \delta(q_0, 1) \cup \delta(q_1, 1) = \{q_0\} \cup \{q_2\} = \{q_0, q_2\}$$

$$\delta^*(q_0, 00) = \bigcup_{s_i \in \delta^*(q_0, 0)} \delta(s_i, 0) = \delta(q_0, 0) \cup \delta(q_1, 0) = \{q_0, q_1\} \cup \emptyset = \{q_0, q_1\}$$

$$\delta^*(q_0, 0) = \bigcup_{s_i \in \delta^*(q_0, \epsilon)} \delta(s_i, 0) = \delta(q_0, 0) = \{q_0, q_1\}$$

$$\delta^*(q_0, \epsilon) = \{q_0\}$$

Language of an NFA

Definition

An NFA $M = (Q, \Sigma, \delta, q_0, F)$ accepts a string w if

$$\delta^*(q_0, w) \cap F \neq \emptyset$$

and the language of M is the set of accepted strings:

$$L(M) = \{w \in \Sigma^* \mid \delta^*(q_0, w) \cap F \neq \emptyset\}$$

Exercises

Design NFAs for the following languages:

1. $L = \{a^n b \mid n \geq 0\}$
2. $L = \{x01y \mid x, y \in \{0,1\}^*\}$
3. $L = \{01w \mid w \in \{0,1\}^*\}$
4. $L = \{w \in \{0,1\}^* \mid w \text{ contains at least two } 0\text{'s}\}$
5. $L = \{w \in \{0,1\}^* \mid w \text{ contains exactly two } 0\text{'s}\}$
6. $L = \{w \in \{0,1\}^* \mid w \text{ has three consecutive } 0\text{'s}\}$

Equivalence of DFA and NFA

Theorem (Equivalence)

A Language L is accepted by some NFA if and only if L is accepted by some DFA.

Proof.

By the two Lemmas below. $\square$

Lemma (DFA to NFA)

Given a DFA D, there always exists an NFA N such that $L(D) = L(N)$.

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DFA to NFA

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Given a DFA D, there always exists an NFA N such that $L(D) = L(N)$.

Proof) Assume a DFA $D = (Q, \Sigma, \delta_D, q_0, F)$ is given. Define an NFA as follows:

$$N = (Q, \Sigma, \delta_N, q_0, F) \text{ where } \delta_N(q, a) = \{\delta_D(q, a)\}$$

To prove:

$$L(D) = \{w \in \Sigma^* \mid \delta_D^*(q_0, w) \in F\} = \{w \in \Sigma^* \mid \delta_N^*(q_0, w) \cap F \neq \emptyset\} = L(N)$$

It is enough to show that

$$\delta_N^*(q_0, w) = \{\delta_D^*(q_0, w)\}$$

The proof is by induction on $|w|$.

- $w = \epsilon$: By the definitions of δ_D^* and δ_N^*, $\delta_D^*(q_0, \epsilon) = q_0$ and $\delta_N^*(q_0, \epsilon) = \{q_0\}$.
- $w = sa$:

$$\begin{aligned}
\delta_N^*(q_0, sa) &= \bigcup_{s_i \in \delta_N^*(q_0, s)} \delta_N(s_i, a) && \text{by definition of } \delta_N^* \\
&= \delta_N(\delta_D^*(q_0, s), a) && \text{by I.H.} \\
&= \{\delta_D(\delta_D^*(q_0, s), a)\} && \text{by definition of } \delta_N \\
&= \{\delta_D^*(q_0, sa)\} && \text{by definition of } \delta_D^*
\end{aligned}$$

NFA to DFA (Subset Construction)

Lemma (NFA to DFA)

Given an NFA N, there always exists a DFA D such that $L(N) = L(D)$.

Proof) Assume an NFA $N = (Q_N, \Sigma, \delta_N, q_0, F_N)$. Define a DFA as follows

$$D = (Q_D, \Sigma, \delta_D, \{q_0\}, F_D)$$

where

- $Q_D = 2^{Q_N}$: A state in the DFA is a set of states of the NFA.
- $F_D = \{S \in Q_D \mid S \cap F_N \neq \emptyset\}$. That is, F_D is all sets of N's states that include at least one final state of N.
- For each $S \in Q_D$ and input symbol $a \in \Sigma$:

$$\delta_D(S, a) = \bigcup_{p \in S} \delta_N(p, a)$$

NFA to DFA

Then, we can prove $L(N) = L(D)$ by showing that

$$\delta_D^*(\{q_0\}, w) = \delta_N^*(q_0, w).$$

The proof is by induction on the length of w.

- $w = \epsilon$: By definition, $\delta_D^*(\{q_0\}, \epsilon) = \{q_0\} = \delta_N^*(q_0, \epsilon)$.
- $w = sa$: Induction hypothesis (I.H.):

$$\delta_D^*(\{q_0\}, s) = \delta_N^*(q_0, s).$$

$$\begin{aligned}
\delta_D^*(\{q_0\}, sa) &= \delta_D(\delta_D^*(\{q_0\}, s), a) && \text{by definition of } \delta_D^* \\
&= \delta_D(\delta_N^*(q_0, s), a) && \text{by I.H.} \\
&= \bigcup_{p \in \delta_N^*(q_0, s)} \delta_N(p, a) && \text{by definition of } \delta_D \\
&= \delta_N^*(q_0, sa) && \text{by definition of } \delta_N^*
\end{aligned}$$

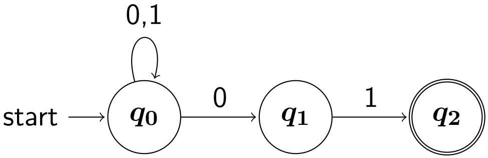
$\square$

Example: Subset Construction

Find a DFA that is equivalent to:

$$N = (\{q_0, q_1, q_2\}, \{0, 1\}, \delta, q_0, \{q_2\})$$

$$\begin{array}{ll} \delta(q_0, 0) = \{q_0, q_1\} & \delta(q_0, 1) = \{q_0\} \\ \delta(q_1, 0) = \emptyset & \delta(q_1, 1) = \{q_2\} \\ \delta(q_2, 0) = \emptyset & \delta(q_2, 1) = \emptyset \end{array}$$



Example: Subset Construction

$$D = (Q_D, \{0,1\}, \delta_d, \{q_0\}, F_D)$$

- $Q_D = 2^{\{q_0,q_1,q_2\}} = \{\emptyset, \{q_0\}, \{q_1\}, \ldots, \{q_0,q_1,q_2\}\}$
- $F_D = \{\{q_2\}, \{q_0,q_2\}, \{q_1,q_2\}, \{q_0,q_1,q_2\}\}$
- δ_D:

	0	1
$\emptyset$	$\emptyset$	$\emptyset$
$\rightarrow \{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_1\}$	$\emptyset$	$\{q_2\}$
$*\{q_2\}$	$\emptyset$	$\emptyset$
$\{q_0, q_1\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$
$*\{q_0, q_2\}$	$\{q_0, q_1\}$	$\{q_0\}$
$*\{q_1, q_2\}$	$\emptyset$	$\{q_2\}$
$*\{q_0, q_1, q_2\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$

Example: Subset Construction