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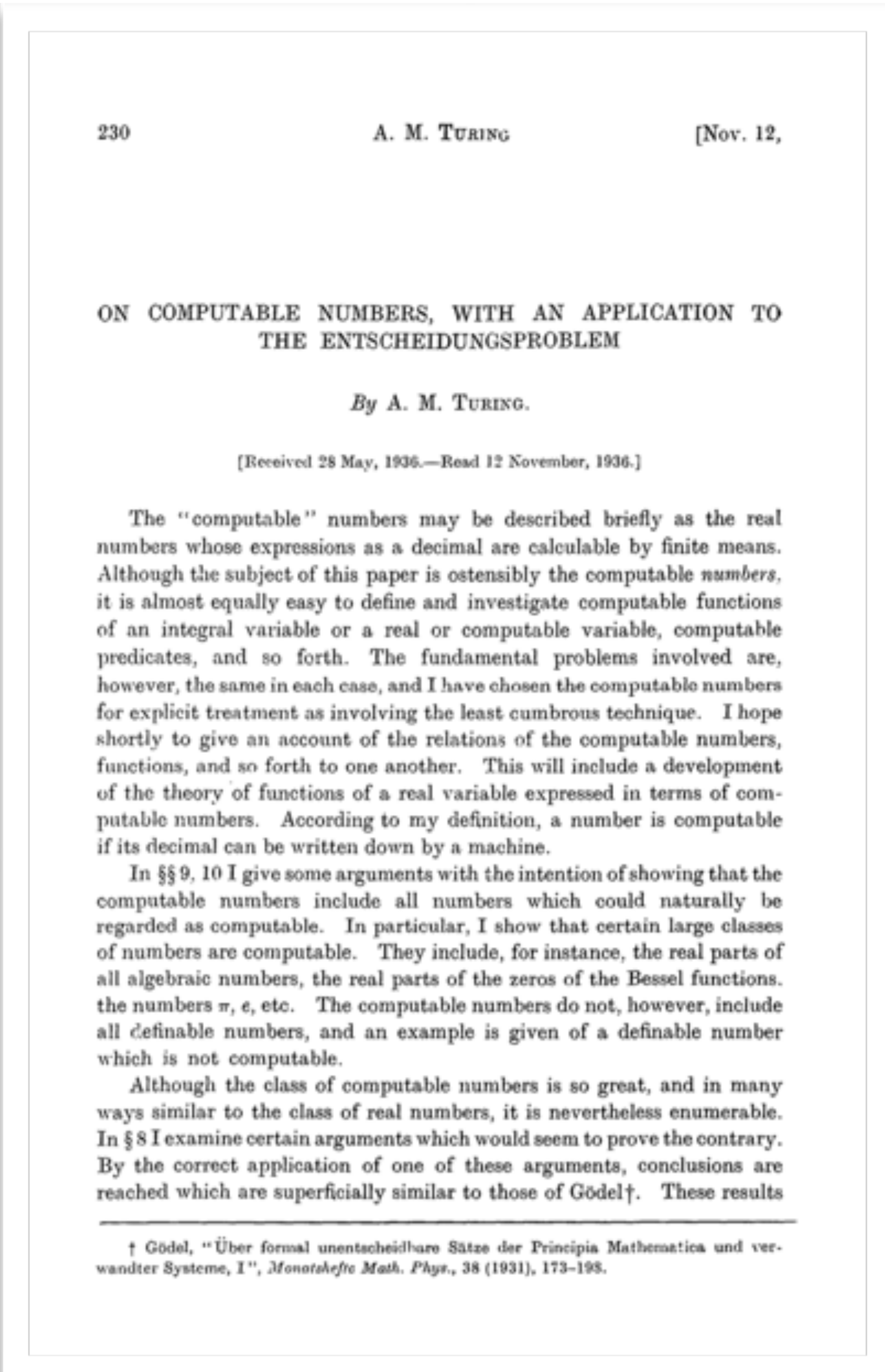

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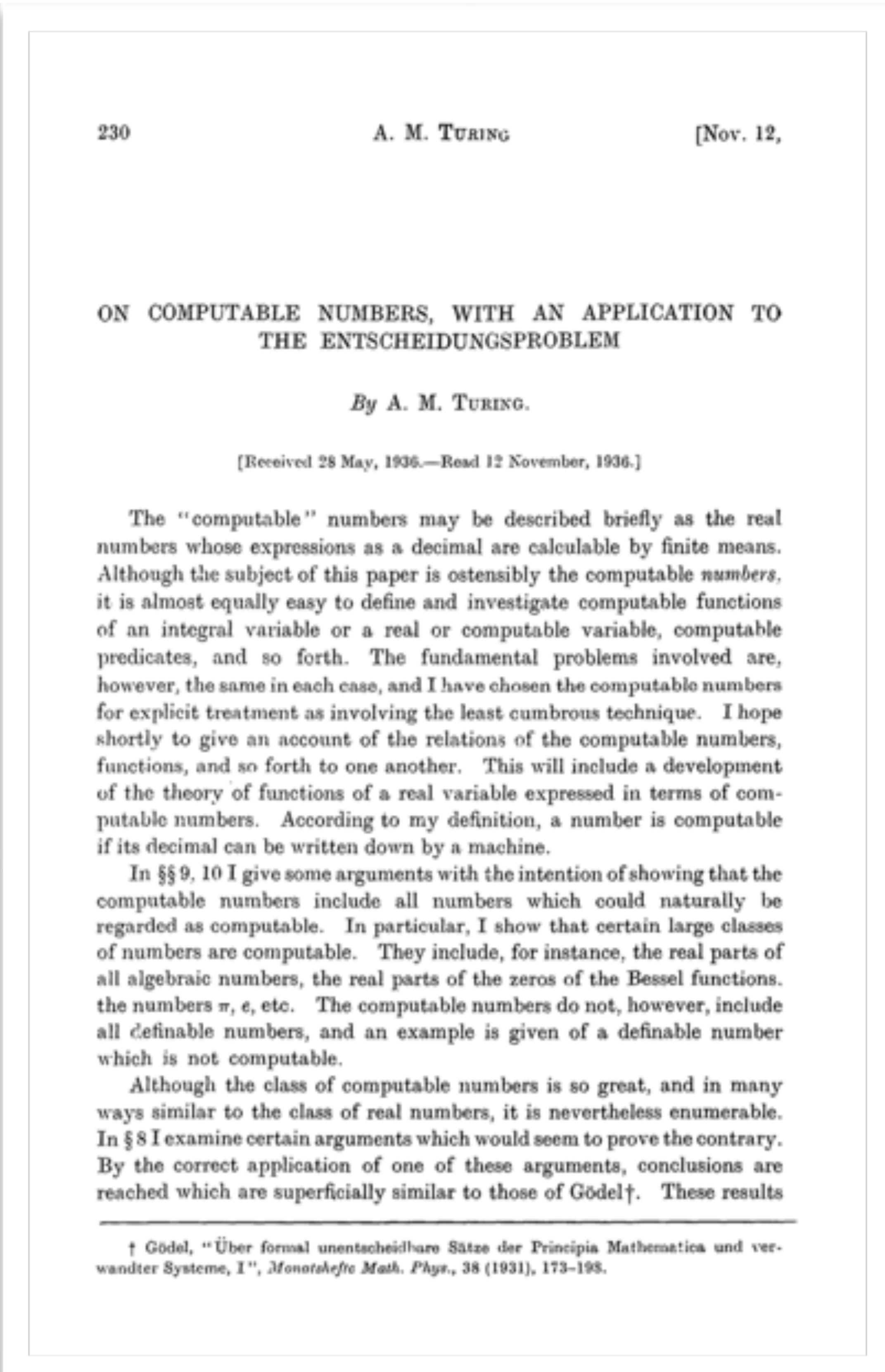

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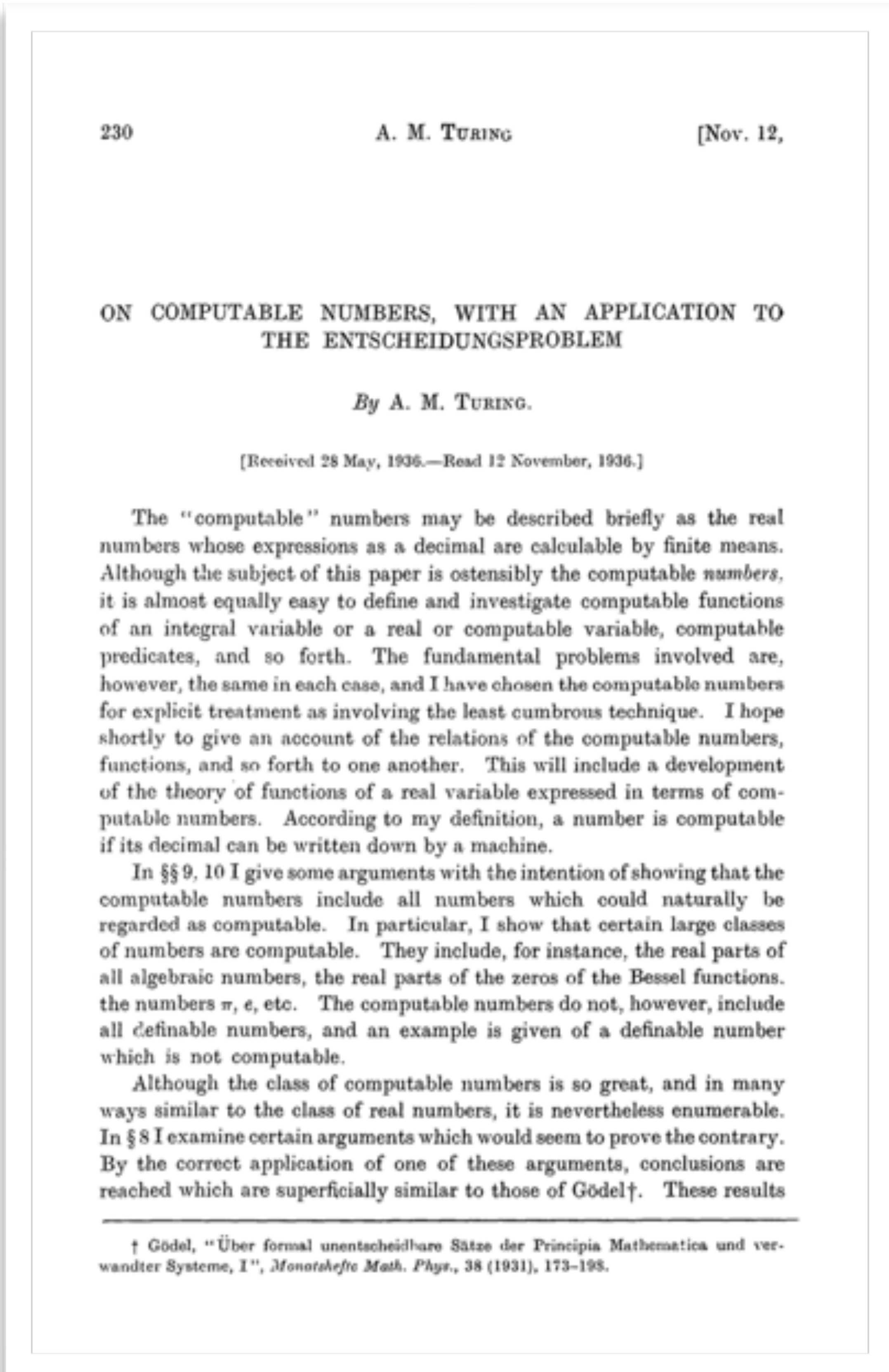

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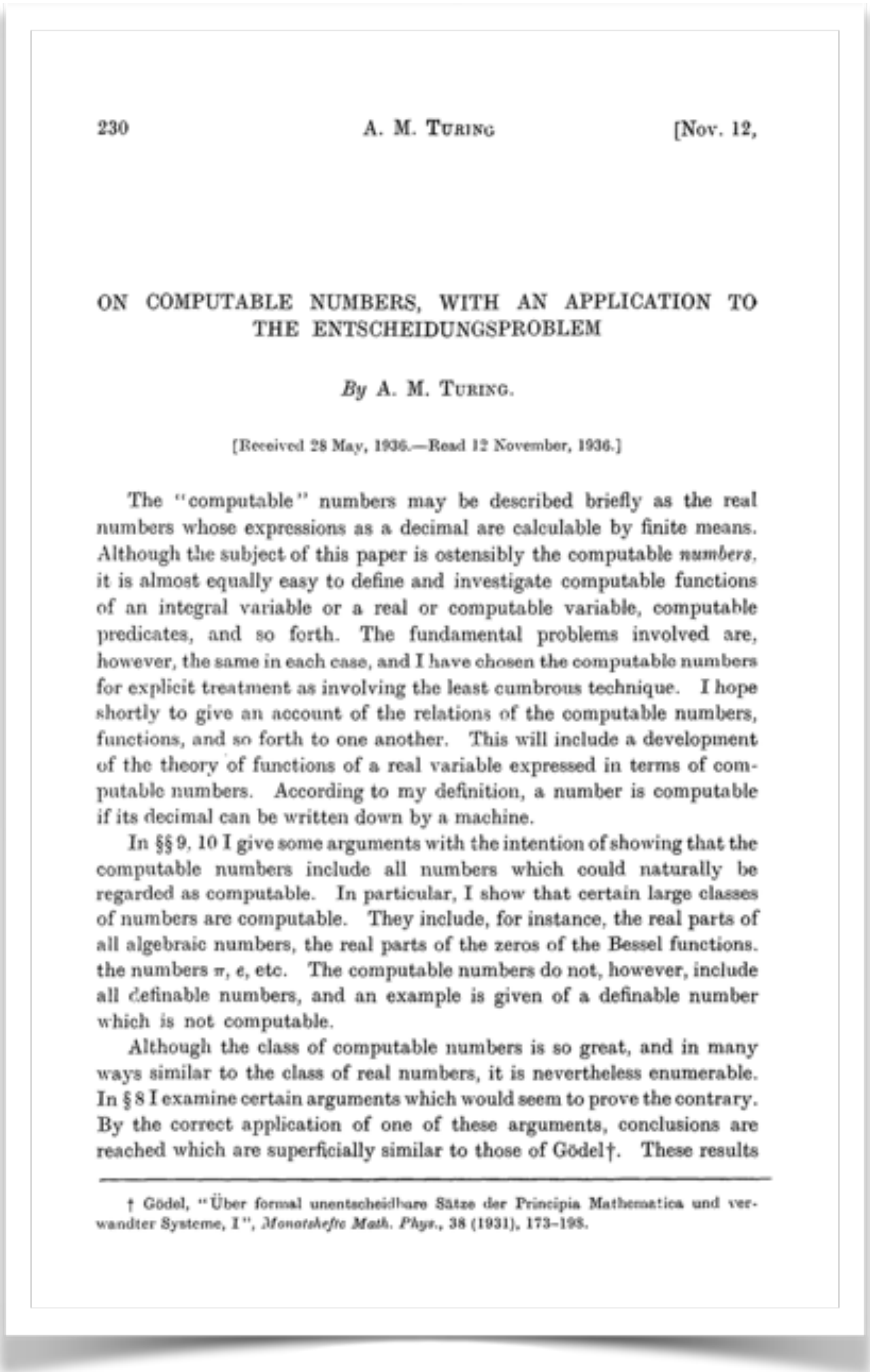
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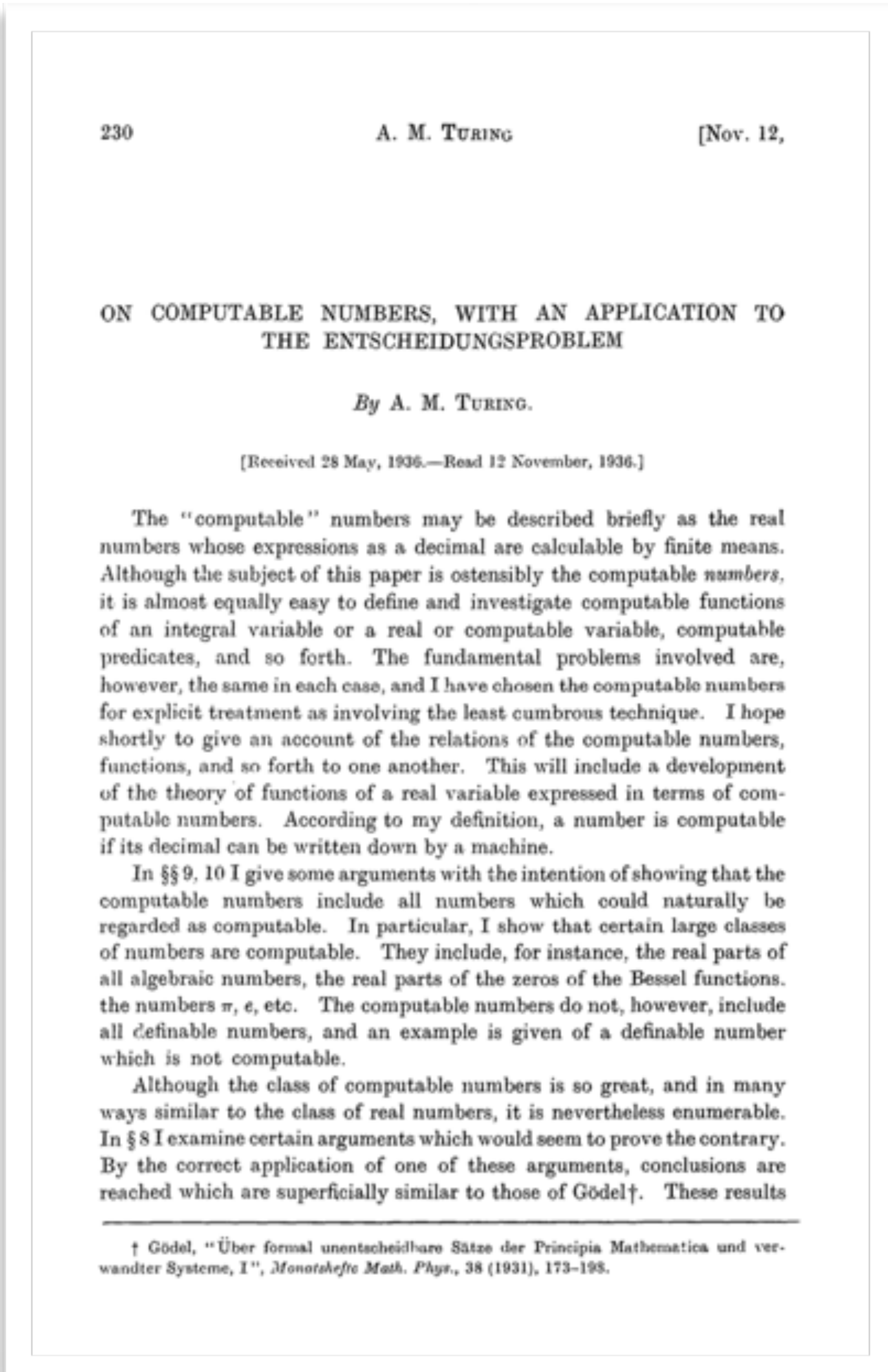

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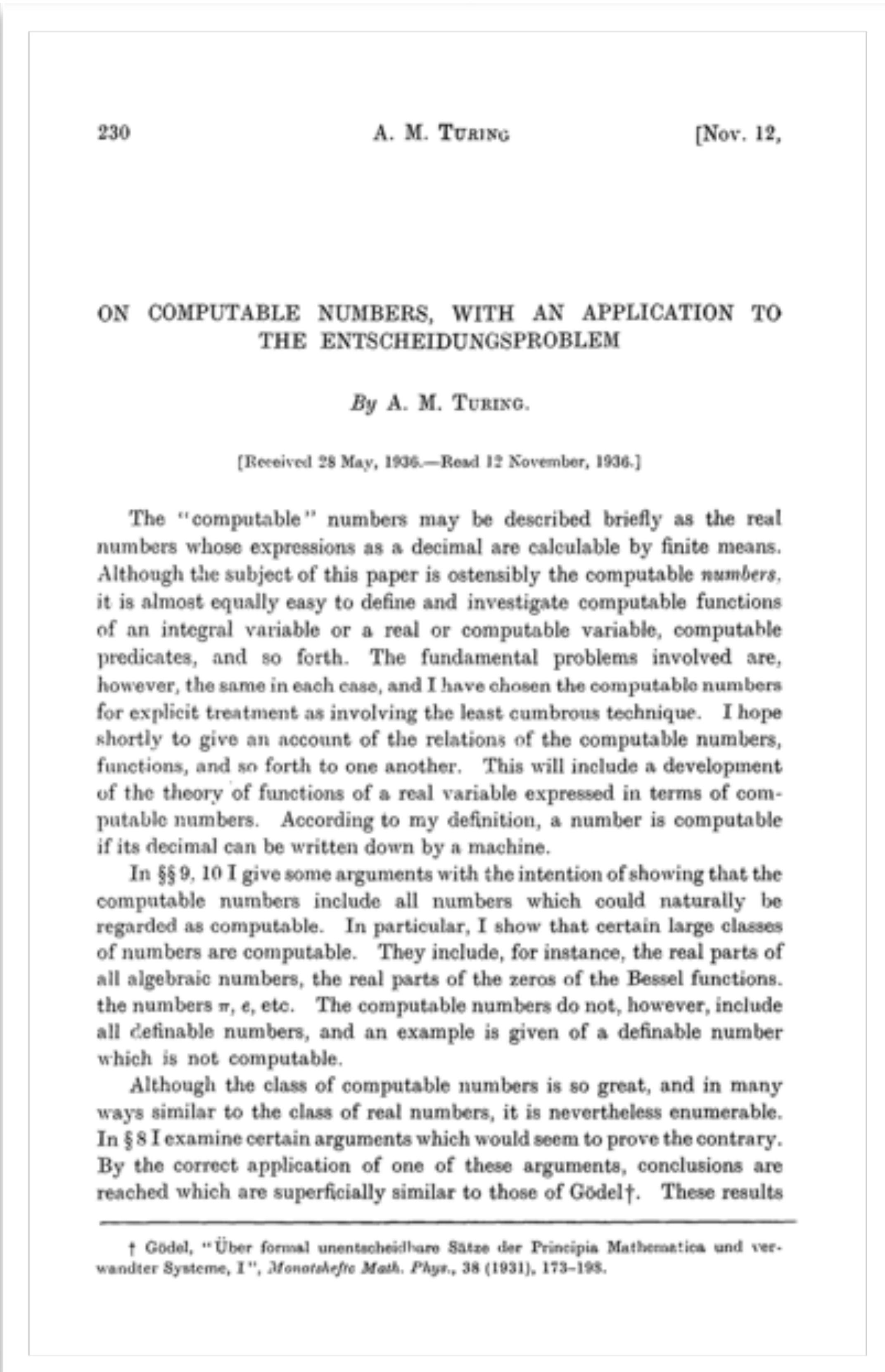

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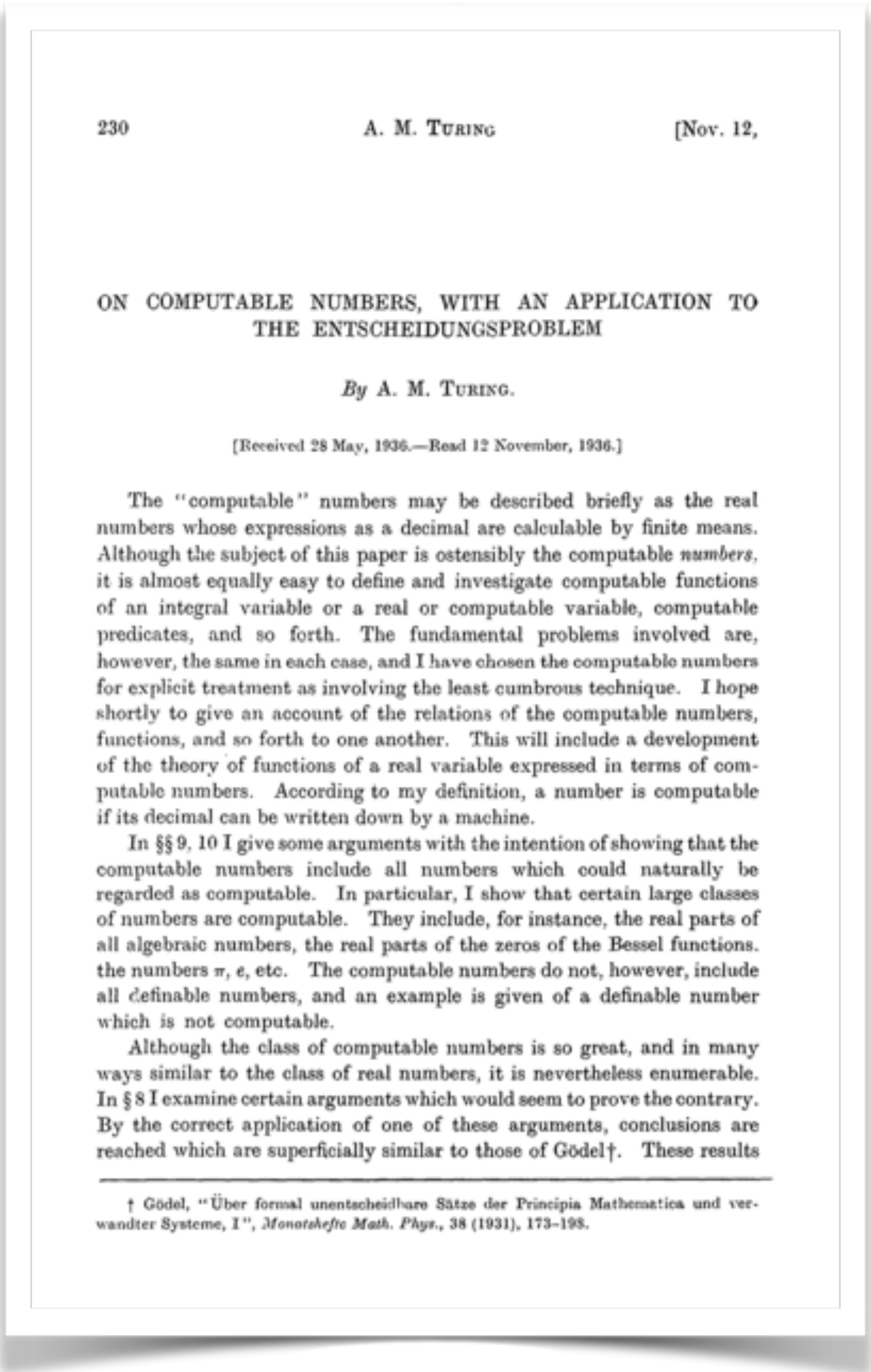
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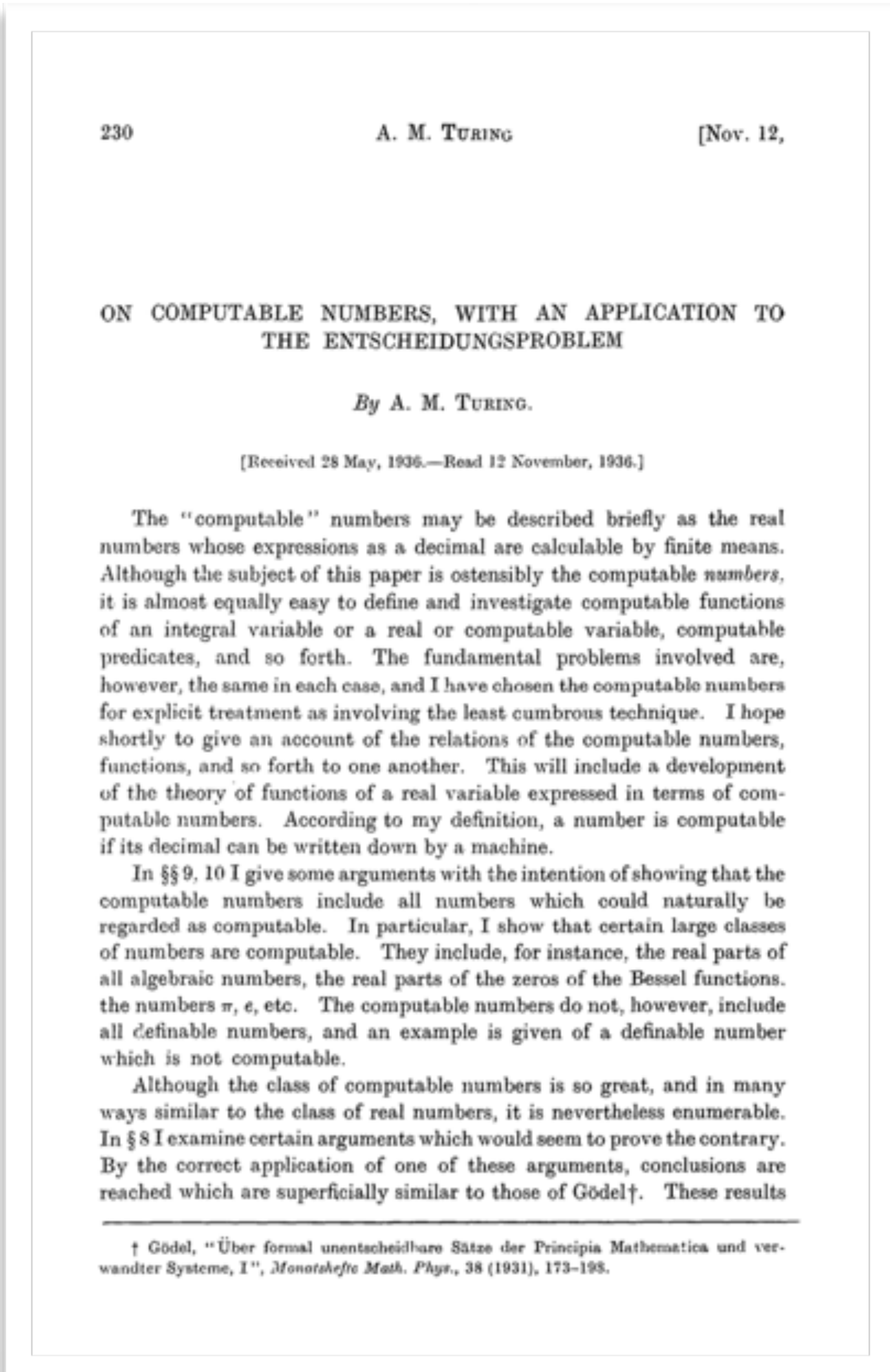

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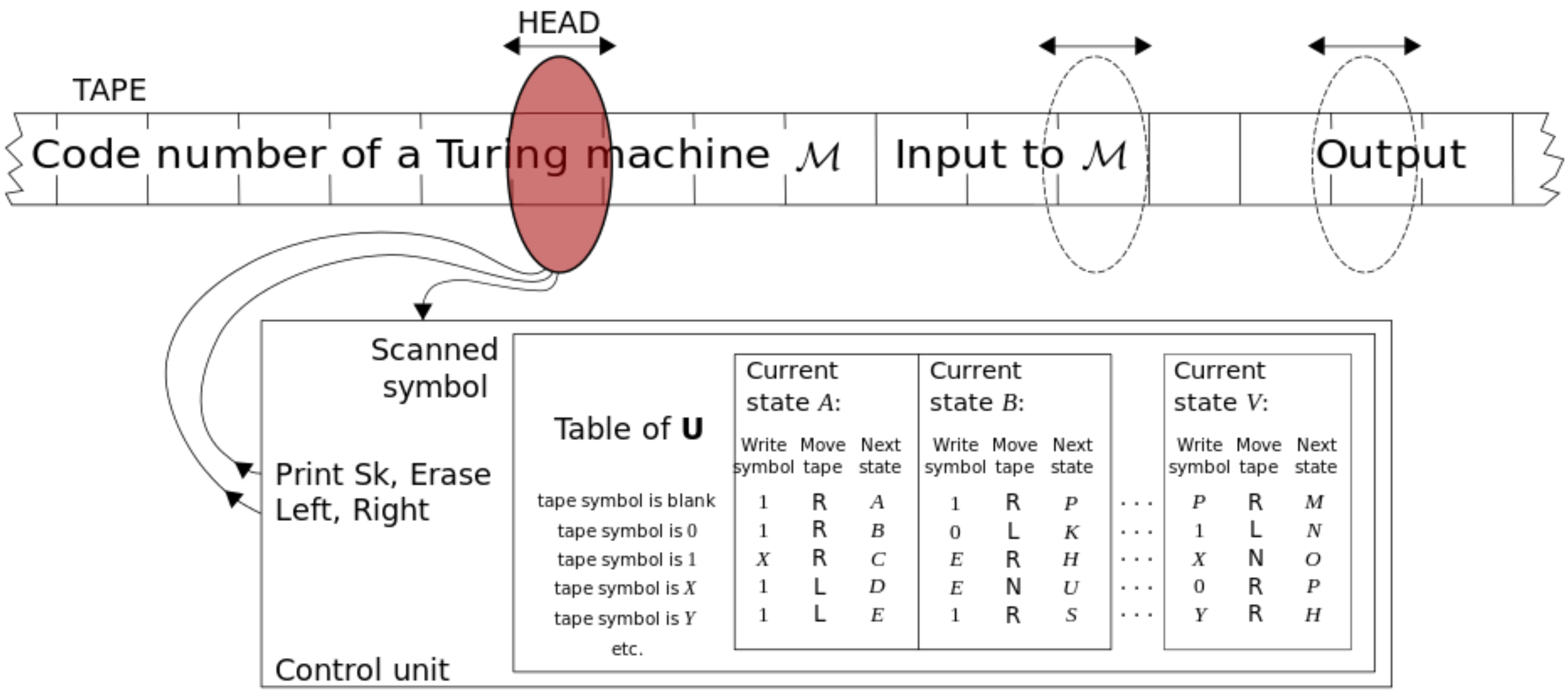
Universal Turing Machine



http://en.wikipedia.org/wiki/Universal_Turing_machine

Can we go beyond TMs?

Trials to extend the Turing machines:

- Stay-option
- Multiple tapes
- Nondeterminism
- …

TM with a Stay-Option

$$(Q, \Sigma, \Gamma, \delta, q_0, B, F)$$

$$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, S\}$$

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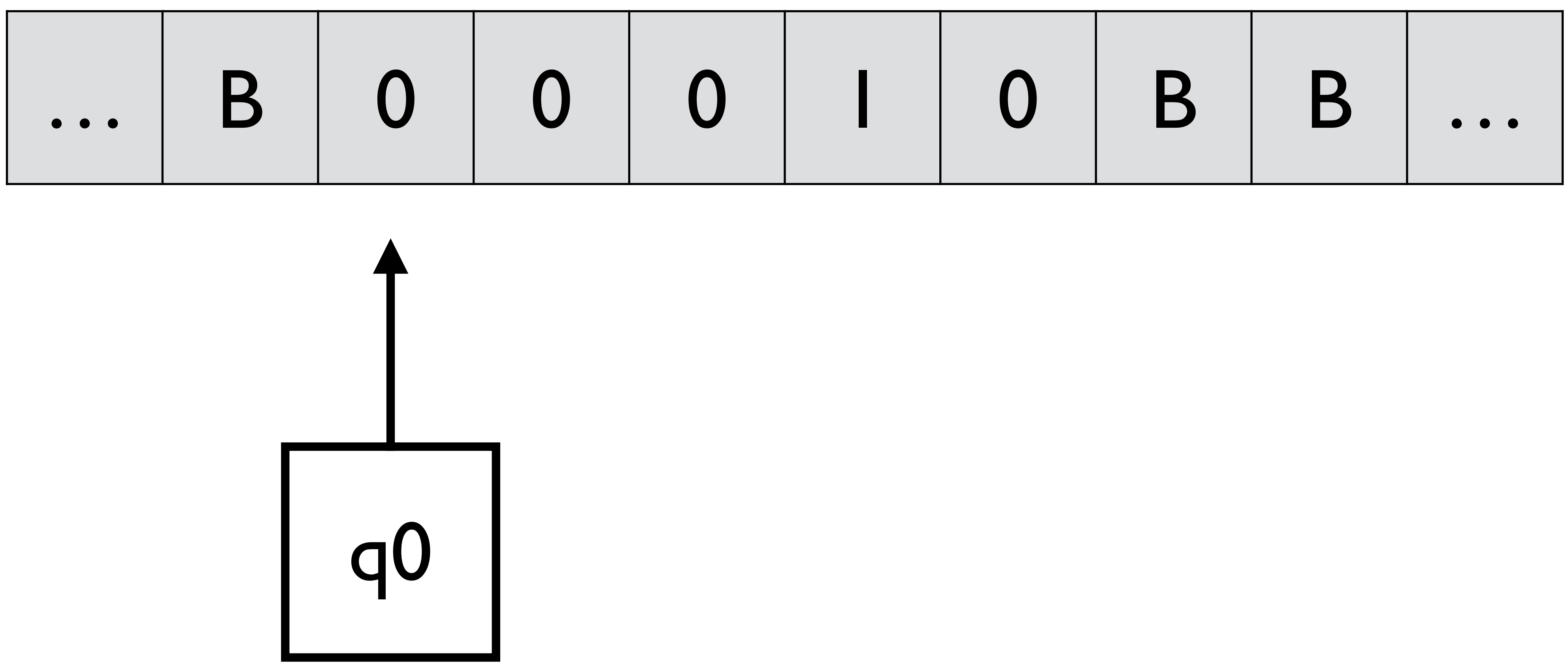
e.g., $\delta(q_0, 0) = (q_1, 1, S)$



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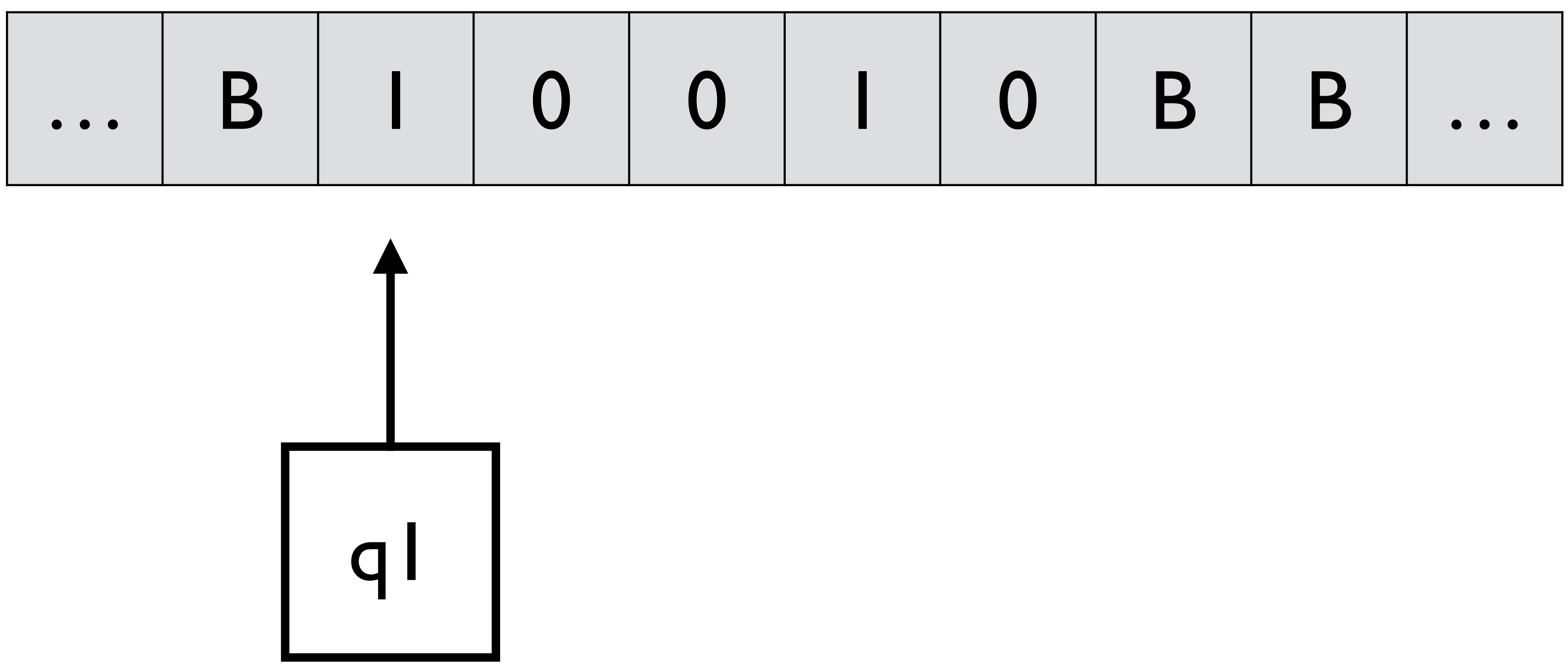


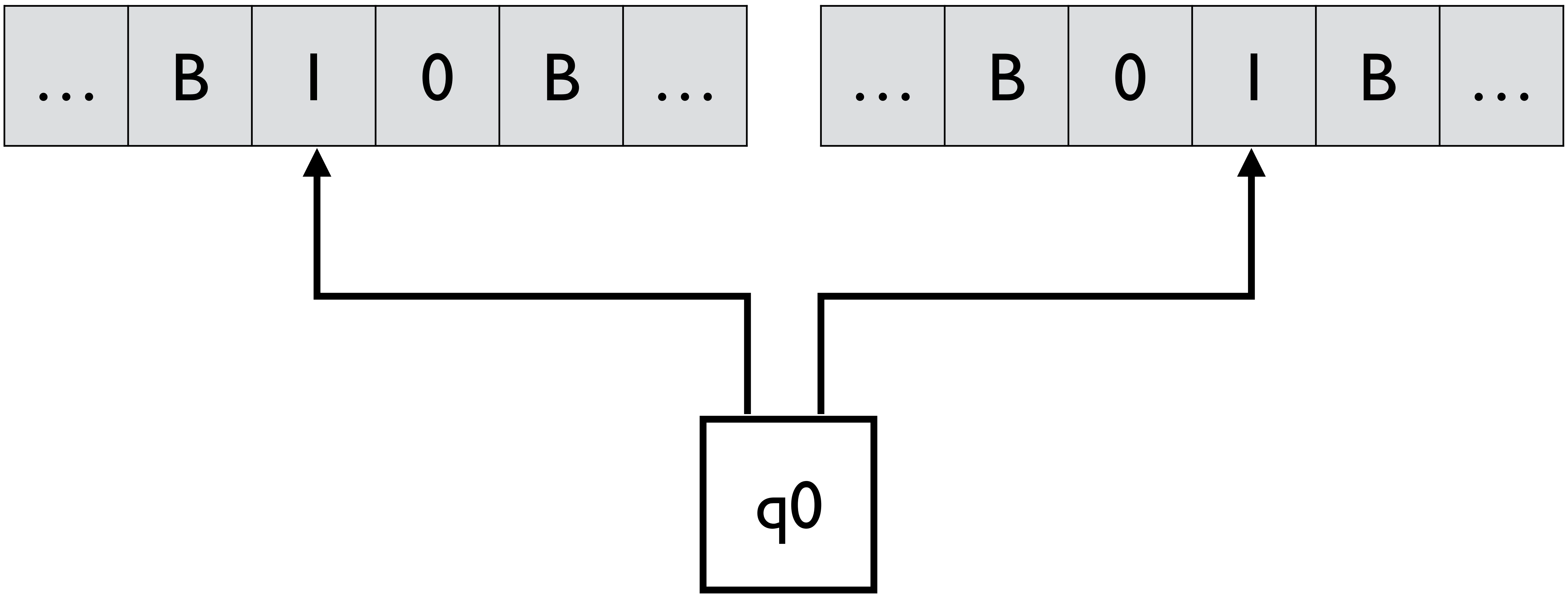
Equivalence

- A Language is accepted by a TM iff it is accepted by a TM/S

Replace $$\delta(q_i, a) = (q_j, b, S)$$

by $$\delta(q_i, a) = (q_k, b, R)$$
$$\delta(q_k, c) = (q_j, c, L)$$

Multitape Turing Machines



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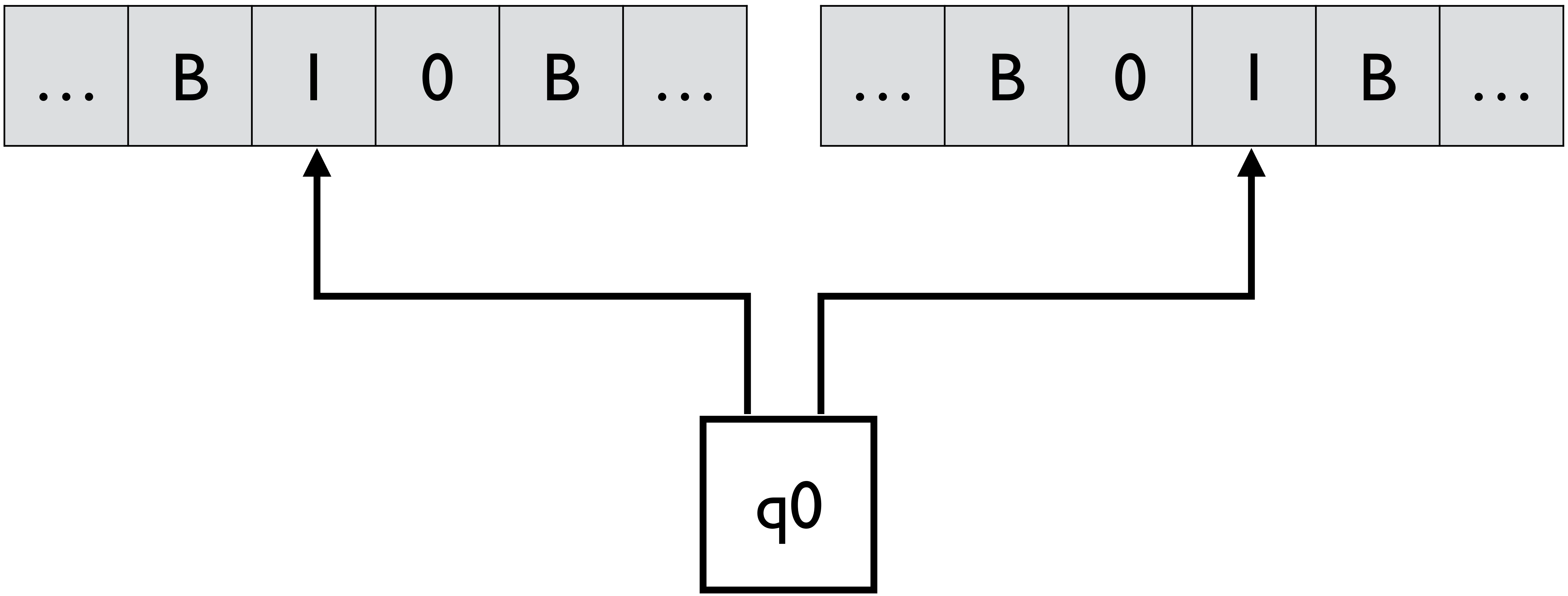

$$(Q, \Sigma, \Gamma, \delta, q_0, B, F)$$

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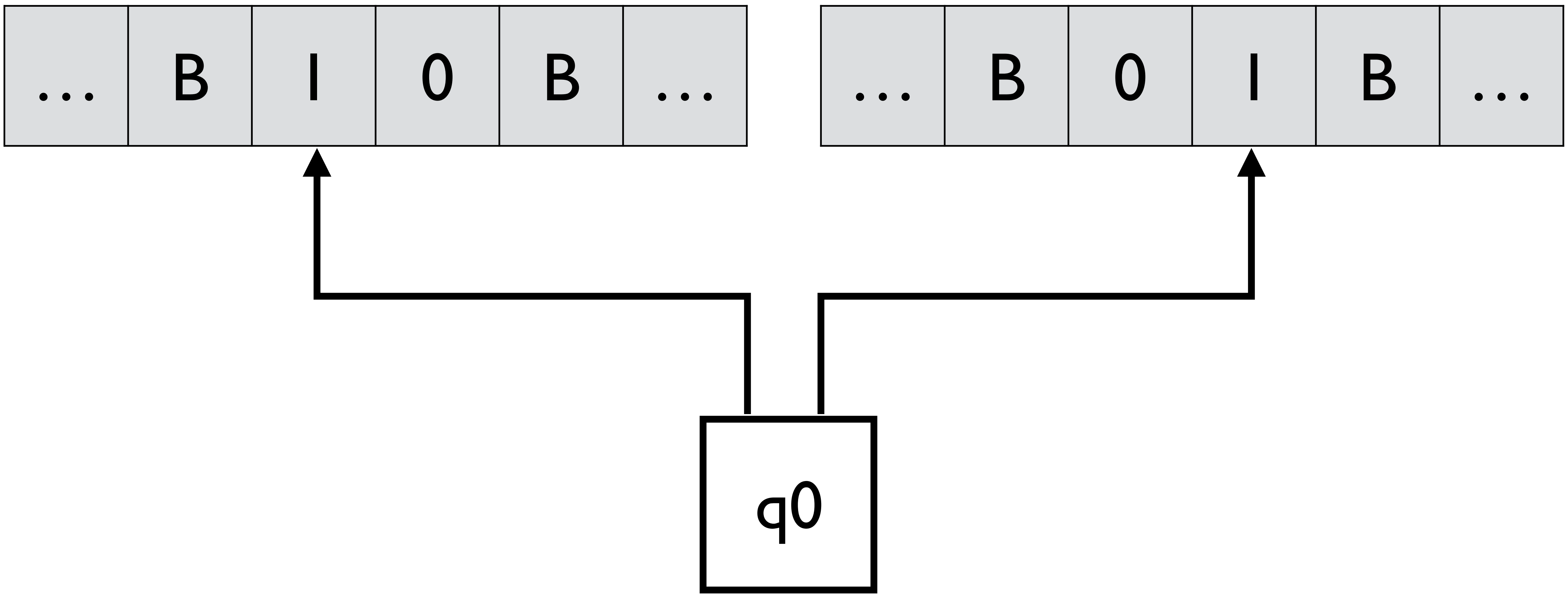

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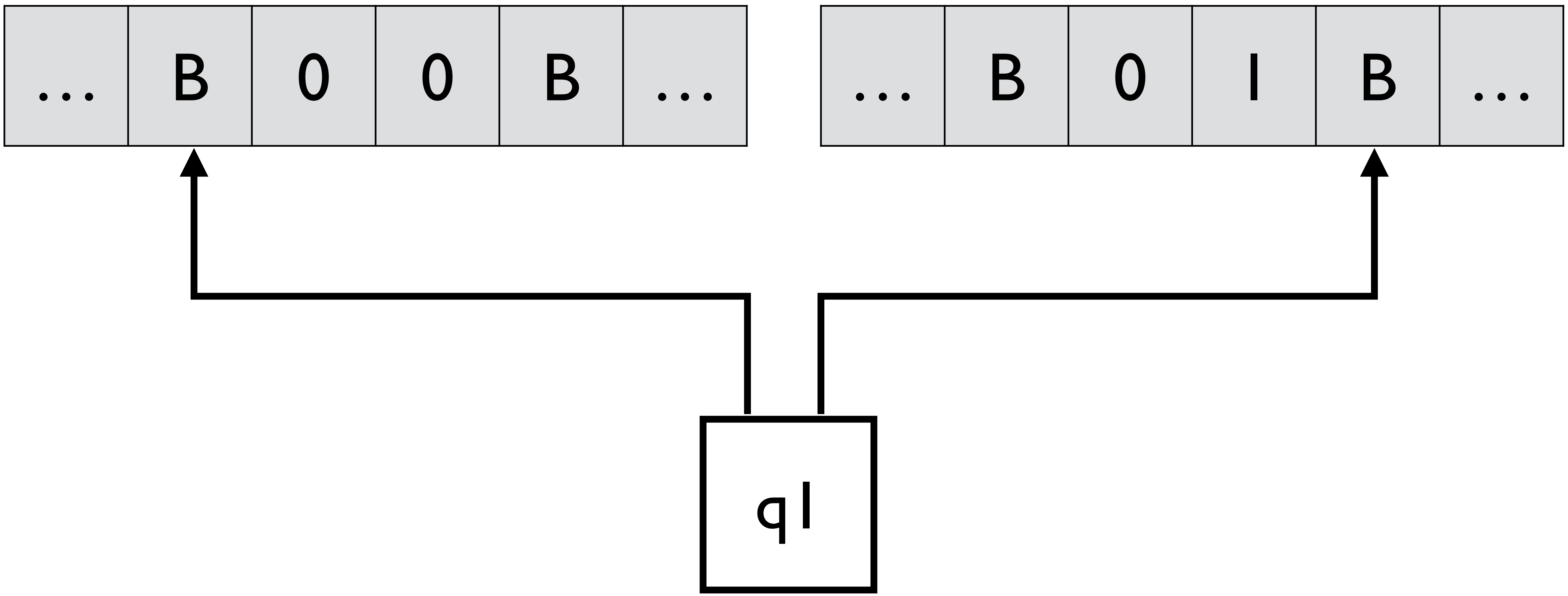

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Equivalence

- Any MTM can be simulated by a standard TM with multiple tracks

...	B	0	0	B	...
...	B	*	B	B	...
...	B	0	1	B	...
...	B	B	*	B	...

cf) Efficiency of MTMs

- MTMs can be more efficient than standard TMs

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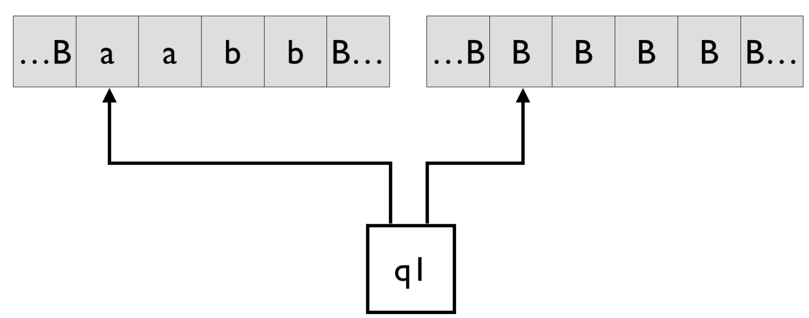
- MTMs can be more efficient than standard TMs

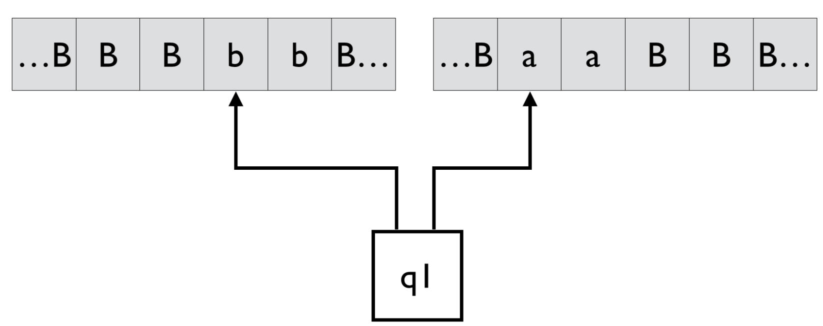
Example

Design a multitape Turing machine that accepts $L = \{a^n b^n \mid n \geq 1\}$.

- In standard TM, repeated back-and-forth movements are required.
- In MTM, copy all a's to tape 2



and then match b's on tape 1 against a's on tape 2



Nondeterministic TMs

$$(Q, \Sigma, \Gamma, \delta, q_0, B, F)$$

$$\delta : Q \times \Gamma \to 2^{Q \times \Gamma \times \{L, R\}}$$

- E.g., $\delta(q_0, a) = \{(q_1, b, R), (q_2, c, L)\}$
- A NTM accepts w if there is a sequence s.t.

$$q_0 w \vdash^* x_1 q_f x_2$$

  with $q_f \in F$.
- Still, equivalent.

cf) Efficiency of NTM

- The equivalent, deterministic TM is exponentially slower than NTM.
- Is this exponential slowdown inevitable? Unknown (P = NP?)

Turing machines are very powerful.

Computable problems are what can be solved by Turing machines
— Turing

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Computable problems are what can be defined by Lambda calculus
— Church

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"Turing-Church Thesis"

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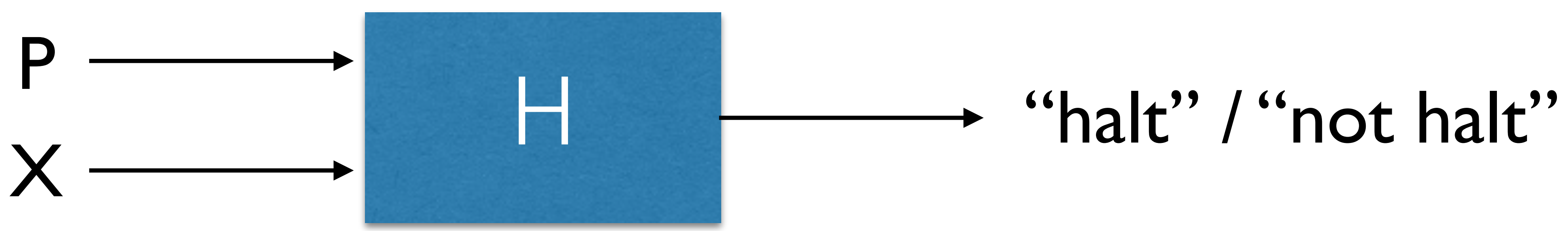
* proof of the existence of incomputable problems:

Halting Problem



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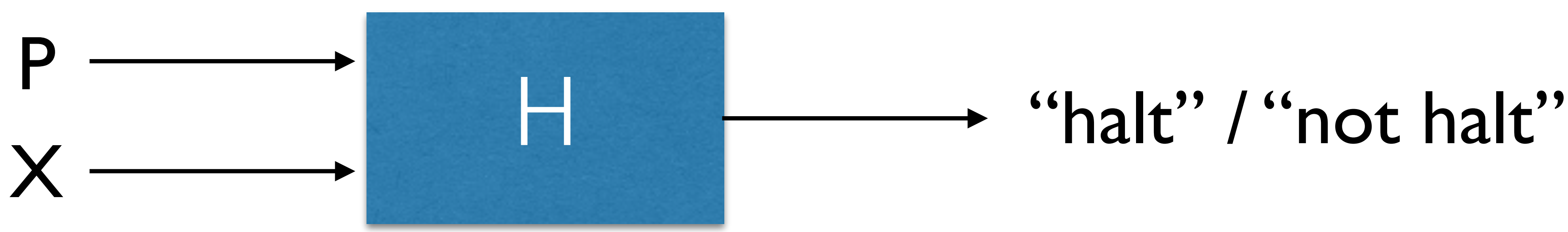
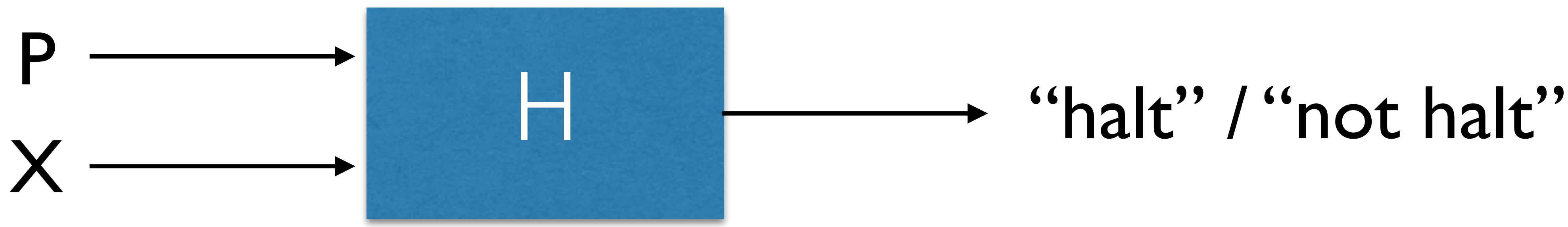


Does such H exist?

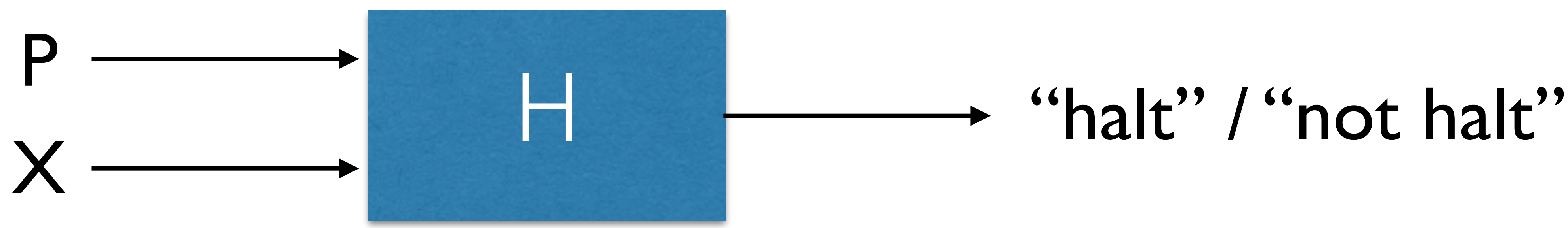
Halting Problem



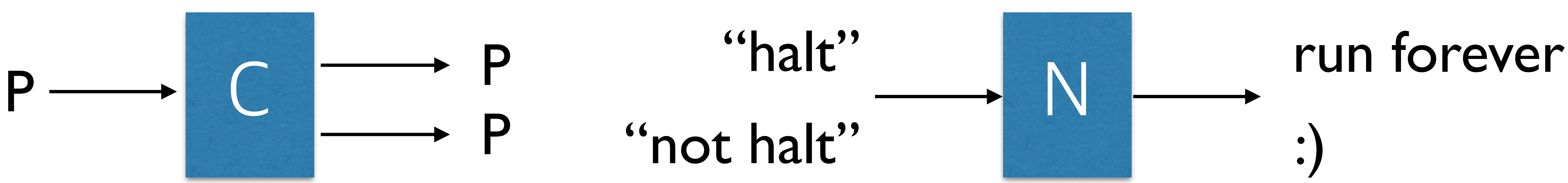
Does such H exist?

No, logically impossible.

Suppose such H exists:



Two simple programs:



Construct the program:

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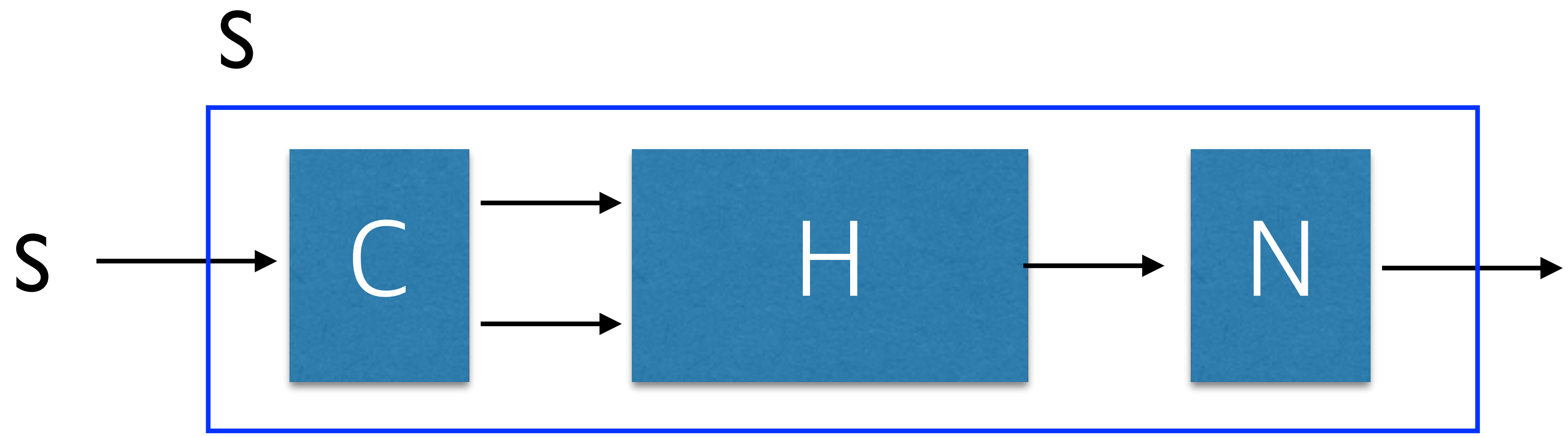

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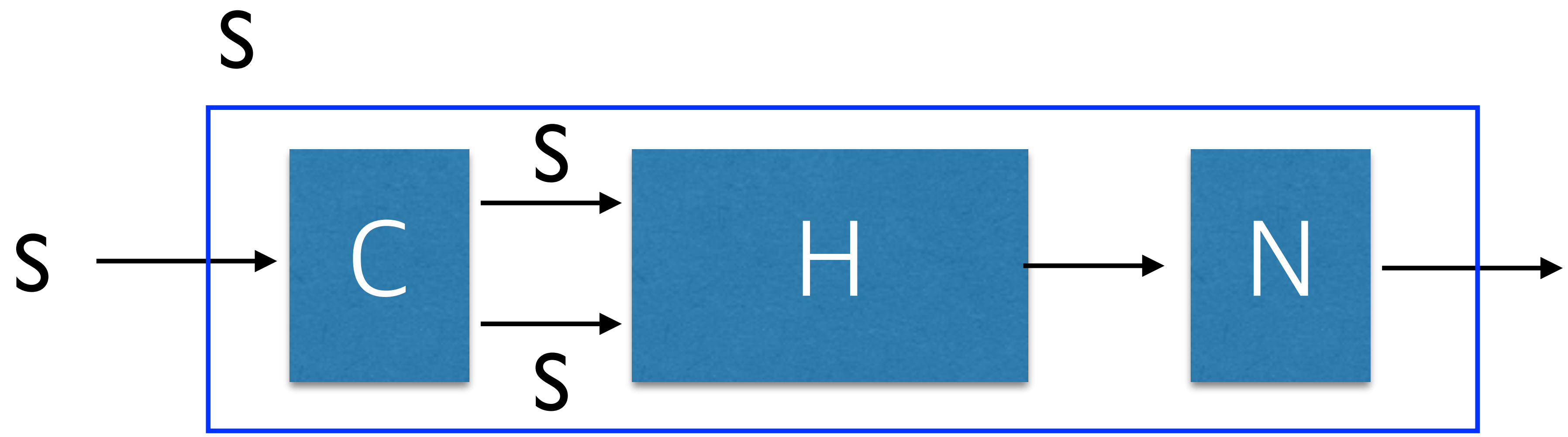

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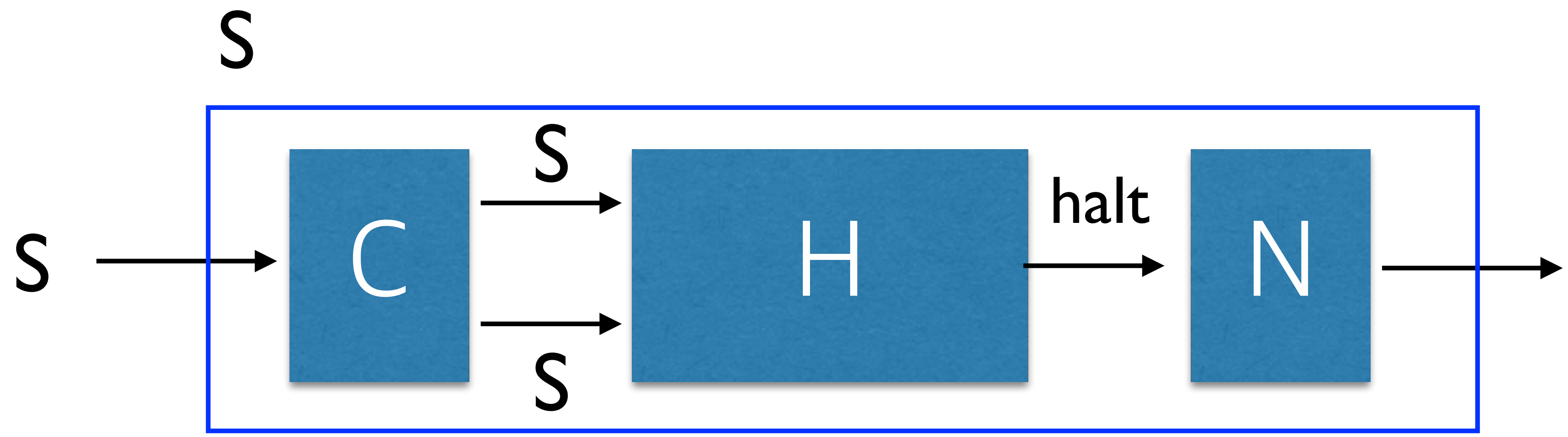

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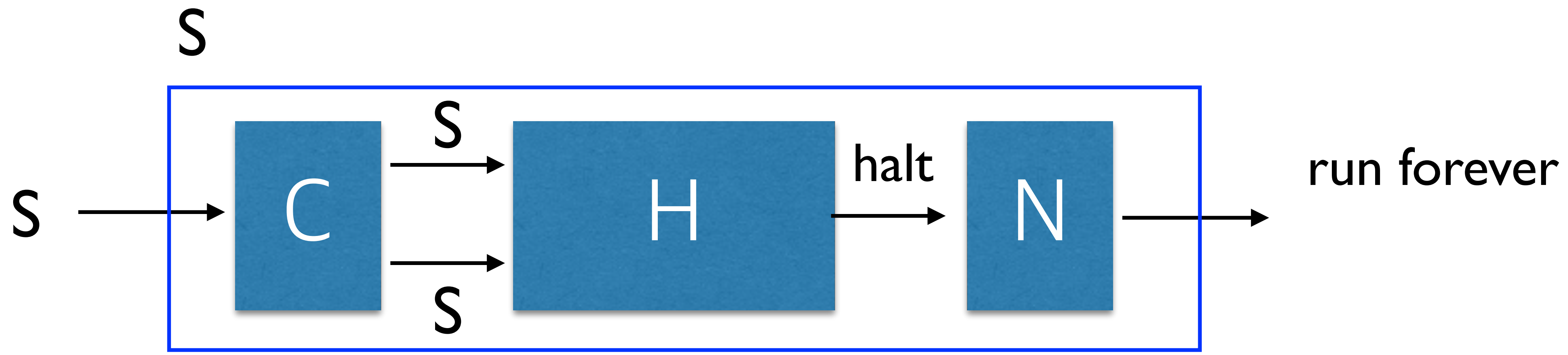
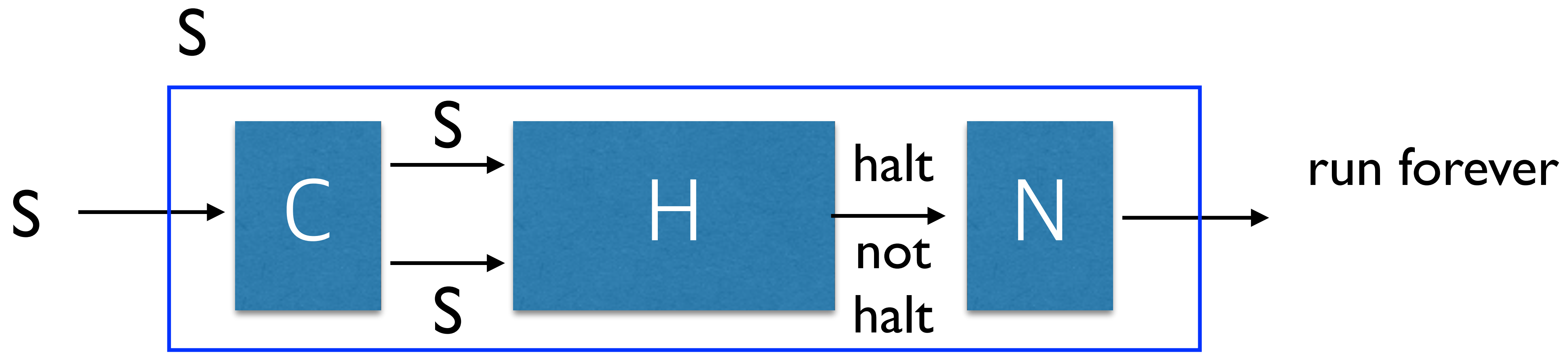


Construct the program:



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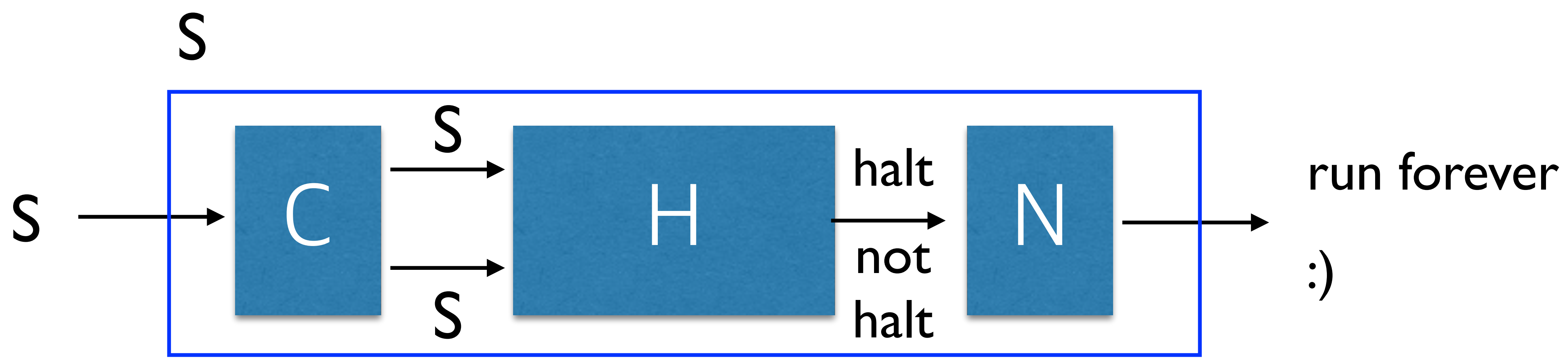

Summary

- Computable problems are what can be solved by Turing machines
- There exist incomputable problems