

COSE212: Programming Languages

Lecture 20 — Lambda Calculus (Origin of Programming Languages)

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A Fundamental Question

Programming languages look very different.

- C, C++, Java, OCaml, Haskell, Scala, JavaScript, etc

Example: QuickSort in C

```
void swap(int* a, int* b) { int t = *a; *a = *b; *b = t; }

int partition (int arr[], int low, int high) {
    int pivot = arr[high];
    int i = (low - 1);

    for (int j = low; j <= high- 1; j++) {
        if (arr[j] <= pivot) {
            i++;
            swap(&arr[i], &arr[j]);
        }
    }
    swap(&arr[i + 1], &arr[high]);
    return (i + 1);
}

void quickSort(int arr[], int low, int high) {
    if (low < high) {
        int pi = partition(arr, low, high);
        quickSort(arr, low, pi - 1);
        quickSort(arr, pi + 1, high);
    }
}
```

Example: QuickSort in Haskell

```
quicksort [] = []  
quicksort (x:xs) = quicksort ys ++ [x] ++ quicksort zs  
    where  
        ys = [a | a <- xs, a <= x]  
        zs = [b | b <- xs, b > x]
```

A Fundamental Question

Are they different fundamentally? or Is there a core mechanism underlying all programming languages?

Syntactic Sugar

- Syntactic sugar is syntax that makes a language “sweet”: it does not add expressiveness but makes programs easier to read and write.
- For example, we can “desugar” the `let` expression:

$$\text{let } x = E_1 \text{ in } E_2 \xRightarrow{\text{desugar}} (\text{proc } x E_2) E_1$$

- Exercise) Desugar the program:

```
let x = 1 in
  let y = 2 in
    x + y
```

Syntactic Sugar

Q) Identify all syntactic sugars of the language:

$$\begin{array}{lcl} E & \rightarrow & n \\ & | & x \\ & | & E + E \\ & | & E - E \\ & | & \text{iszero } E \\ & | & \text{if } E \text{ then } E \text{ else } E \\ & | & \text{let } x = E \text{ in } E \\ & | & \text{letrec } f(x) = E \text{ in } E \\ & | & \text{proc } x E \\ & | & E E \end{array}$$

Lambda Calculus (λ -Calculus)

- By removing all syntactic sugars from the language, we obtain a minimal language, called *lambda calculus*:

$$\begin{array}{lll} e & \rightarrow & x \quad \text{variables} \\ & | & \lambda x.e \quad \text{abstraction} \\ & | & e \ e \quad \text{application} \end{array}$$

Programming language = Lambda calculus + Syntactic sugars

Origins of Programming Languages and Computer

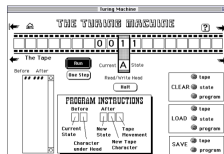


- In 1935, Church developed λ -calculus as a formal system for mathematical logic and argued that any computable function on natural numbers can be computed with λ -calculus. Since then, λ -calculus became the model of programming languages.
- In 1936, Turing independently developed Turing machine and argued that any computable function on natural numbers can be computed with the machine. Since then, Turing machine became the model of computers.

Church-Turing Thesis

- A surprising fact is that the classes of λ -calculus and Turing machines can compute coincide even though they were developed independently.
- Church and Turing proved that the classes of computable functions defined by λ -calculus and Turing machine are equivalent.

$$\begin{array}{ccc} e & \rightarrow & x \\ | & & \lambda x.e \\ | & & e e \end{array} =$$



A function is λ -computable if and only if Turing computable.

- This equivalence has led mathematicians and computer scientists to believe that these models are “universal”: A function is computable if and only if λ -computable if and only if Turing computable.

λ -Calculus is Everywhere

λ -calculus had immense impacts on programming languages.

- It has been the core of functional programming languages (e.g., Lisp, ML, Haskell, Scala, etc).
- Lambdas in other languages:

- ▶ Java8

```
(int n, int m) -> n + m
```

- ▶ C++11

```
[](int x, int y) { return x + y; }
```

- ▶ Python

```
(lambda x, y: x + y)
```

- ▶ JavaScript

```
function (a, b) { return a + b }
```

Syntax of Lambda Calculus

e	$\rightarrow$	x	variables
		$\lambda x.e$	abstraction
		$e e$	application

- Examples:

$$\begin{array}{ccccc} & & x & y & z \\ & & \lambda x.x & \lambda x.y & \lambda x.\lambda y.x \\ x\ y & (\lambda x.x)\ z & x\ \lambda y.z & ((\lambda x.x)\ \lambda x.x) \end{array}$$

- Conventions when writing λ -expressions:

- 1 Application associates to the left, e.g., $s\ t\ u = (s\ t)\ u$
- 2 The body of an abstraction extends as far to the right as possible, e.g., $\lambda x.\lambda y.x\ y\ x = \lambda x.(\lambda y.((x\ y)\ x))$

Bound and Free Variables

- An occurrence of variable x is said to be *bound* when it occurs inside λx , otherwise said to be *free*.
 - ▶ $\lambda y.(x\ y)$
 - ▶ $\lambda x.x$
 - ▶ $\lambda z.\lambda x.\lambda x.(y\ z)$
 - ▶ $(\lambda x.x)\ x$
- Expressions without free variables is said to be *closed expressions* or *combinators*.

Evaluation

To evaluate λ -expression e ,

- 1 Find a sub-expression of the form:

$$(\lambda x. e_1) e_2$$

Expressions of this form are called “redex” (reducible expression).

- 2 Rewrite the expression by substituting the e_2 for every free occurrence of x in e_1 :

$$(\lambda x. e_1) e_2 \rightarrow [x \mapsto e_2] e_1$$

This rewriting is called β -reduction

Repeat the above two steps until there are no redexes.

Evaluation

- $\lambda x.x$
- $(\lambda x.x) y$
- $(\lambda x.x y)$
- $(\lambda x.x y) z$
- $(\lambda x.(\lambda y.x)) z$
- $(\lambda x.(\lambda x.x)) z$
- $(\lambda x.(\lambda y.x)) y$
- $(\lambda x.(\lambda y.x y)) (\lambda x.x) z$

Substitution

The definition of $[x \mapsto e_1]e_2$:

$$\begin{aligned}[x \mapsto e_1]x &= e_1 \\[x \mapsto e_1]y &= y \\[x \mapsto e_1](\lambda y. e_2) &= \lambda z. [x \mapsto e_1]([y \mapsto z]e_2) \quad (\text{new } z) \\[x \mapsto e_1](e_2 e_3) &= ([x \mapsto e_1]e_2 [x \mapsto e_1]e_3)\end{aligned}$$

Evaluation Strategy

- In a lambda expression, multiple redexes may exist. Which redex to reduce next?

$$\lambda x.x (\lambda x.x (\lambda z.(\lambda x.x) z)) = id (id (\lambda z.id z))$$

redexes:

$$\frac{id (id (\lambda z.id z))}{id (id (\lambda z.id z))}$$
$$\frac{id (id (\lambda z.id z))}{id (id (\lambda z.\underline{id} z))}$$

- Evaluation strategies:
 - ▶ Normal order
 - ▶ Call-by-name
 - ▶ Call-by-value

Normal order strategy

Reduce the leftmost, outermost redex first:

$$\begin{aligned} & id (id (\lambda z.id z)) \\ \rightarrow & \frac{id (id (\lambda z.id z))}{id (\lambda z.id z)} \\ \rightarrow & \lambda z.id z \\ \rightarrow & \lambda z.z \\ \not\rightarrow & \end{aligned}$$

The evaluation is deterministic (i.e., partial function).

Call-by-name strategy

Follow the normal order reduction, not allowing reductions inside abstractions:

$$\begin{aligned} & id (id (\lambda z.id z)) \\ \rightarrow & \frac{id (id (\lambda z.id z))}{id (\lambda z.id z)} \\ \rightarrow & \lambda z.id z \\ \not\rightarrow & \end{aligned}$$

The call-by-name strategy is *non-strict* (or *lazy*) in that it evaluates arguments that are actually used.

Call-by-value strategy

Reduce the outermost redex whose right-hand side has a *value* (a term that cannot be reduced any further):

$$\begin{aligned} & id \ (id \ (\lambda z.id \ z)) \\ \rightarrow & \ \underline{id \ (\lambda z.id \ z)} \\ \rightarrow & \ \lambda z.id \ z \\ \nrightarrow & \end{aligned}$$

The call-by-name strategy is *strict* in that it always evaluates arguments, whether or not they are used in the body.

Compiling to Lambda Calculus

Consider the source language:

$$\begin{array}{lcl} E & \rightarrow & \textit{true} \\ & | & \textit{false} \\ & | & n \\ & | & x \\ & | & E + E \\ & | & \textit{iszero } E \\ & | & \textit{if } E \textit{ then } E \textit{ else } E \\ & | & \textit{let } x = E \textit{ in } E \\ & | & \textit{letrec } f(x) = E \textit{ in } E \\ & | & \textit{proc } x \ E \\ & | & E \ E \end{array}$$

Define the translation procedure from E to λ -calculus.

Compiling to Lambda Calculus

E : the translation result of E in λ -calculus

$$\underline{true} = \lambda t. \lambda f. t$$

$$\underline{false} = \lambda t. \lambda f. f$$

$$\underline{0} = \lambda s. \lambda z. z$$

$$\underline{1} = \lambda s. \lambda z. (s\ z)$$

$$\underline{n} = \lambda s. \lambda z. (s^n\ z)$$

$$\underline{x} = x$$

$$\underline{E_1 + E_2} = (\lambda n. \lambda m. \lambda s. \lambda z. m\ s\ (n\ s\ z))\ \underline{E_1}\ \underline{E_2}$$

$$\underline{\text{iszero } E} = (\lambda m. m\ (\lambda x. \underline{false})\ \underline{true})\ \underline{E}$$

$$\underline{\text{if } E_1 \text{ then } E_2 \text{ else } E_3} = \underline{E_1}\ \underline{E_2}\ \underline{E_3}$$

$$\underline{\text{let } x = E_1 \text{ in } E_2} = (\lambda x. \underline{E_2})\ \underline{E_1}$$

$$\underline{\text{letrec } f(x) = E_1 \text{ in } E_2} = \underline{\text{let } f = Y\ (\lambda f. \lambda x. E_1) \text{ in } E_2}$$

$$\underline{\text{proc } x\ E} = \underline{\lambda x. E}$$

$$\underline{E_1\ E_2} = \underline{E_1}\ \underline{E_2}$$

Correctness of Compilation

Theorem

For any expression E ,

$$\llbracket \underline{E} \rrbracket = \underline{\llbracket E \rrbracket}$$

where $\llbracket E \rrbracket$ denotes the value that results from evaluating E .

Examples: Booleans

$$\begin{aligned}\underline{\text{if } \textit{true} \text{ then } 0 \text{ else } 1} &= \underline{\textit{true}} \underline{0} \underline{1} \\ &= (\lambda t. \lambda f. t) \underline{0} \underline{1} \\ &= \underline{0} \\ &= \lambda s. \lambda z. z\end{aligned}$$

Note that

$$\llbracket \underline{\text{if } \textit{true} \text{ then } 0 \text{ else } 1} \rrbracket = \llbracket \underline{\text{if } \textit{true} \text{ then } 0 \text{ else } 1} \rrbracket$$

Exercises

Define the translation for the boolean operations:

- E_1 and E_2 =
- E_1 or E_2 =
- not E =

Example: Numerals

$$\begin{aligned}\underline{1} + \underline{2} &= (\lambda n. \lambda m. \lambda s. \lambda z. m \ s \ (n \ s \ z)) \ \underline{1} \ \underline{2} \\ &= \lambda s. \lambda z. \underline{2} \ s \ (\underline{1} \ s \ z) \\ &= \lambda s. \lambda z. \underline{2} \ s \ (\lambda s. \lambda z. (s \ z) \ s \ z) \\ &= \lambda s. \lambda z. \underline{2} \ s \ (s \ z) \\ &= \lambda s. \lambda z. (\lambda s. \lambda z. (s \ (s \ z))) \ s \ (s \ z) \\ &= \lambda s. \lambda z. s \ (s \ (s \ z)) \\ &= \underline{3}\end{aligned}$$

Exercises

Define the translation for the boolean operations:

- $\text{succ } \underline{E} =$

- $\text{pred } \underline{E} =$

- $\underline{E_1 * E_2} =$

- $\underline{E_1^{E_2}} =$

Recursion

- For example, the factorial function

$$f(n) = \text{if } n = 0 \text{ then } 1 \text{ else } n * f(n - 1)$$

is encoded by

$$\text{fact} = Y(\lambda f. \lambda n. \text{if } n = 0 \text{ then } 1 \text{ else } n * f(n - 1))$$

where Y is the Y-combinator (or fixed point combinator):

$$Y = \lambda f. (\lambda x. f (x x)) (\lambda x. f (x x))$$

- Then, $\text{fact } n$ computes $n!$.
- Recursive functions can be encoded by composing non-recursive functions!

Recursion

Let $F = \lambda f. \lambda n. \text{if } n = 0 \text{ then } 1 \text{ else } n * f(n - 1)$ and $G = \lambda x. F(x \ x)$.

$$\begin{aligned} & \text{fact } 1 \\ &= (Y \ F) \ 1 \\ &= (\lambda f. ((\lambda x. f(x \ x))(\lambda x. f(x \ x)))) \ F) \ 1 \\ &= ((\lambda x. F(x \ x))(\lambda x. F(x \ x))) \ 1 \\ &= (G \ G) \ 1 \\ &= (F \ (G \ G)) \ 1 \\ &= (\lambda n. \text{if } n = 0 \text{ then } 1 \text{ else } n * (G \ G)(n - 1)) \ 1 \\ &= \text{if } 1 = 0 \text{ then } 1 \text{ else } 1 * (G \ G)(1 - 1)) \\ &= \text{if false then } 1 \text{ else } 1 * (G \ G)(1 - 1)) \\ &= 1 * (G \ G)(1 - 1) \\ &= 1 * (F \ (G \ G))(1 - 1) \\ &= 1 * (\lambda n. \text{if } n = 0 \text{ then } 1 \text{ else } n * (G \ G)(n - 1))(1 - 1) \\ &= 1 * \text{if } (1 - 1) = 0 \text{ then } 1 \text{ else } (1 - 1) * (G \ G)((1 - 1) - 1) \\ &= 1 * 1 \end{aligned}$$

Summary

Programming language = Lambda calculus + Syntactic sugars

- λ -calculus is a minimal programming language.
 - ▶ Syntax: $e \rightarrow x \mid \lambda x.e \mid e e$
 - ▶ Semantics: β -reduction
- Yet, λ -calculus is Turing-complete.

$$\begin{array}{c} e \rightarrow x \\ | \lambda x.e \\ | e e \end{array} =$$

