

COSE212: Programming Languages

Lecture 18 — Let-Polymorphic Type System

Hakjoo Oh
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Motivation

- Our type system is useful but it is not as expressive as we would like it to be. In particular, it does not support *polymorphism*¹. For example, it rejects the following program:

```
let f = proc (x) x in
  if (f (iszero (0))) then (f 11) else (f 22)
```

- Polymorphic functions are widely used in practice, so OCaml supports polymorphism:

```
# let f = fun x -> x in
  if (f (0=0)) then (f 11) else (f 22);;
- : int = 11
```

- Let's extend our type system to the let-polymorphic type system, the ML-style polymorphism.

¹Polymorphism refers to the language mechanisms that allow a single part of a program to be used with different types in different contexts

What went wrong?

```
let f = proc (x) x in  
  if (f (iszero (0))) then (f 11) else (f 22)
```

- We assign type $t \rightarrow t$ to f , generating the constraint that the argument and return types are the same.
- Intuitively, the program can be well typed because the all usages of f satisfy the required constraint:
 - ▶ In $(f \text{ (iszero 0)})$, we can assign $\text{bool} \rightarrow \text{bool}$ to f .
 - ▶ In $(f \text{ 11})$ and $(f \text{ 22})$, we can assign $\text{int} \rightarrow \text{int}$ to f .
- However, our type checking algorithm uses the same type variable t in both cases and generates the spurious constraint that $\text{bool} = \text{int}$.
- Any idea to fix this problem?

A Simple Solution

Associate a *different* variable t with each use of f . This is easily accomplished by substituting the body of f for each occurrence of f . For example, convert the program

```
let f = proc (x) x in
  if (f (iszero (0))) then (f 11) else (f 22)
```

into the following before type-checking:

```
if ((proc (x) x) (iszero (0)))
then ((proc (x) x) 11)
else ((proc (x) x) 22)
```

which is accepted by our type system as we can generate different type variables for different copies of the procedure.

Typing Rule

Instead of the ordinary typing rule for let:

$$\frac{\Gamma \vdash E_1 : t_1 \quad [x \mapsto t_1]\Gamma \vdash E_2 : t_2}{\Gamma \vdash \text{let } x = E_1 \text{ in } E_2 : t_2}$$

we use the new typing rule:

$$\frac{\Gamma \vdash [x \mapsto E_1]E_2 : t_2}{\Gamma \vdash \text{let } x = E_1 \text{ in } E_2 : t_2}$$

The corresponding algorithm for generating type equation:

$$\mathcal{V}(\Gamma, \text{let } x = e_1 \text{ in } e_2, t) = \mathcal{V}(\Gamma, [x \mapsto e_1]e_2, t)$$

The ordinary unification algorithm does the rest.

Flaws

This simplistic method has some flaws that need to be addressed before we can use it in practice.

- 1 Unused definitions are not type-checked, so a program like

```
let x = <unsafe code> in 5
```

will pass the type-checker. (This can be easily fixed. See Exercise 1)

- 2 The method is not efficient if the body of `let` contains many occurrences of the bound variables:

```
let a = <complex code> in
```

```
  let b = a + a in
```

```
    let c = b + b in
```

```
      let d = c + c in
```

```
        ...
```

The typing rule can cause the type-checker to perform an amount of work that is exponential in the size of the original code.

Exercise 1

Fix the typing rule and $\mathcal{V}$ to repair the first problem.

Let-Polymorphic Type Checking Algorithm

To avoid the re-computation, practical implementations of languages with let-polymorphism use a more clever algorithm. In outline, the type-checking of

$$\text{let } x = e_1 \text{ in } e_2$$

proceeds as follows:

- We find the most general type t of e_1 by running the ordinary type-checking algorithm.
- We *generalize* any variables remaining in the type, obtaining the *type scheme* $\forall \alpha_1 \dots \alpha_n. t$, where $\alpha_1 \dots \alpha_n$ appear in t .
- We extend the type environment to record the type scheme for the bound variable x , and start type-checking e_2
- Each time we encounter an occurrence of x , we generate fresh type variables $\beta_1 \dots \beta_n$ and use them to instantiate the type scheme.

Example 1

$\text{let } f = \text{proc } (x) \ 1 \text{ in } (f \ 1) + (f \ \text{true})$

Example 2

```
let  $f$  = proc ( $x$ )  $x$  if ( $f$  true) then 1 else (( $f$   $f$ ) 2)
```

Generalization Is Not Always Safe

Care is needed when generalizing types because doing so is not always safe. For example, consider the program:

```
proc (c)
  (let f = proc (x) c in
    if (f true) then 1 else ((f f) 2))
```

- The most general type for f is $t_1 \rightarrow t_2$.
- Generalizing the type, we obtain the type scheme $\forall t_1, t_2. t_1 \rightarrow t_2$.
- The body of `let` is well-typed by instantiating t_2 to `bool` for the first occurrence of f and to some function type for the second occurrence of f . The type system accepts the program.
- However, the program produces runtime error because no value c can be both a boolean and a procedure.
- To fix this problem, we disallow generalization for any type variables that are mentioned in the type environment. The safe type scheme for f is $\forall t_1. t_1 \rightarrow t_2$. With this generalization the program gets rejected.

Let-Polymorphic Type System

Types and Type Schemes:

$$\begin{aligned} t &::= \text{int} \mid \text{bool} \mid t \rightarrow t \mid \alpha \\ \sigma &::= t \mid \forall \bar{\alpha}. t \end{aligned}$$

Type Environment:

$$\Gamma : Id \rightarrow \sigma$$

Free type variables, Generalization, Instantiation:

$$\mathbf{FTV}(\Gamma) = \bigcup_{x \in \text{Dom}(\Gamma)} \mathbf{FTV}(\Gamma(x)), \quad \mathbf{Gen}(\Gamma, t) = \forall (\mathbf{FTV}(t) \setminus \mathbf{FTV}(\Gamma)). t$$

$$\mathbf{Inst}(\forall \bar{\alpha}. t) = t[\bar{\alpha} \mapsto \bar{\beta}] \text{ with fresh } \bar{\beta} \text{ and } \mathbf{Inst}(t) = t$$

Typing Rules (selected)

$$\frac{x : \sigma \in \Gamma \quad \text{Inst}(\sigma) = t}{\Gamma \vdash x : t} \text{VAR}$$

$$\frac{\Gamma \vdash e_1 : t_1 \quad \sigma = \mathbf{Gen}(\Gamma, t_1) \quad [x \mapsto \sigma]\Gamma \vdash e_2 : t_2}{\Gamma \vdash \text{let } x = e_1 \text{ in } e_2 : t_2} \text{LET}$$

$$\frac{[x \mapsto \sigma]\Gamma \vdash e : t_2}{\Gamma \vdash \text{proc}(x) e : t_1 \rightarrow t_2} \text{PROC} \qquad \frac{\Gamma \vdash e_1 : t_2 \rightarrow t \quad \Gamma \vdash e_2 : t_2}{\Gamma \vdash e_1 e_2 : t} \text{APP}$$

$$\frac{\Gamma \vdash e_1 : \text{bool} \quad \Gamma \vdash e_2 : t \quad \Gamma \vdash e_3 : t}{\Gamma \vdash \text{if } e_1 \text{ then } e_2 \text{ else } e_3 : t} \text{IF}$$

Efficiency

- The algorithm is much more efficient than the simplistic approach.
- In practice, its time complexity is almost linear.
- However, the worst-case time complexity is still exponential.
- For example, try to evaluate the following OCaml program. It takes a very long time to typecheck.

```
let f0 = fun x -> (x,x) in
  let f1 = fun y -> f0 (f0 y) in
    let f2 = fun y -> f1 (f1 y) in
      let f3 = fun y -> f2 (f2 y) in
        let f4 = fun y -> f3 (f3 y) in
          let f5 = fun y -> f4 (f4 y) in
            f5 (fun z -> z)
```

Summary

- We extended our type system (called *simple type system*) to *let-polymorphic type system*, the core of ML type system.
- The extension is conservative:

$$\Gamma \vdash_{\text{simple}} E : T \implies \Gamma \vdash_{\text{poly}} E : T$$

Let-polymorphic type system accepts all programs acceptable by the simple type system.