COSE212: Programming Languages

Lecture 15 — Automatic Type Inference (3)

Hakjoo Oh
2025 Fall

Finding a Solution of Type Equations

Find the values of type variables that make all the equations true.

$$\underbrace{\texttt{proc}\ (\underbrace{f}_{t_f})\ \underbrace{\texttt{proc}\ (\underbrace{x}_{t_x})\ \underbrace{((\underbrace{f\ 3)}_{t_3} - (\underbrace{f\ x}_{t_4}))}_{t_2}}_{t_1}}_{t_0}$$

Equations			Solution		
t_0	$=$	$t_f \to t_1$	t_0	$=$	$(\text{int} \to \text{int}) \to (\text{int} \to \text{int})$
t_1	$=$	$t_x \to t_2$	t_1	$=$	$\text{int} \to \text{int}$
t_3	$=$	int	t_2	$=$	int
t_4	$=$	int	t_3	$=$	int
t_2	$=$	int	t_4	$=$	int
t_f	$=$	$\text{int} \to t_3$	t_f	$=$	$\text{int} \to \text{int}$
t_f	$=$	$t_x \to t_4$	t_x	$=$	int

Static type systems find such a solution using *unification algorithm*.

Example 1

The calculation is split into equations to be solved and substitution found so far. Initially, the substitution is empty:

Equations	Substitution
$t_0 = t_f \rightarrow t_1$	
$t_1 = t_x \rightarrow t_2$	
$t_3 = \mathsf{int}$	
$t_4 = \mathsf{int}$	
$t_2 = \mathsf{int}$	
$t_f = \mathsf{int} \rightarrow t_3$	
$t_f = t_x \rightarrow t_4$	

Example 1

Consider each equation in turn. If the equation's left-hand side is a variable, move it to the substitution:

Equations	Substitution
$t_1 = t_x \to t_2$	$t_0 = t_f \to t_1$
$t_3 = \mathsf{int}$	
$t_4 = \mathsf{int}$	
$t_2 = \mathsf{int}$	
$t_f = \mathsf{int} \to t_3$	
$t_f = t_x \to t_4$	

Example 1

Move the next equation to the substitution and propagate the information to the existing substitution (i.e., substitute the right-hand side for each occurrence of t_1):

Equations	Substitution
$t_3 = \text{int}$	$t_0 = t_f \rightarrow (t_x \rightarrow t_2)$
$t_4 = \text{int}$	$t_1 = t_x \rightarrow t_2$
$t_2 = \text{int}$	
$t_f = \text{int} \rightarrow t_3$	
$t_f = t_x \rightarrow t_4$	

Example 1

Same for the next three equations:

Equations	Substitution
$t_4 = \text{int}$	$t_0 = t_f \to (t_x \to t_2)$
$t_2 = \text{int}$	$t_1 = t_x \to t_2$
$t_f = \text{int} \to t_3$	$t_3 = \text{int}$
$t_f = t_x \to t_4$	

Equations	Substitution
$t_2 = \text{int}$	$t_0 = t_f \to (t_x \to t_2)$
$t_f = \text{int} \to t_3$	$t_1 = t_x \to t_2$
$t_f = t_x \to t_4$	$t_3 = \text{int}$
	$t_4 = \text{int}$

Equations	Substitution
$t_f = \text{int} \to t_3$	$t_0 = t_f \to (t_x \to \text{int})$
$t_f = t_x \to t_4$	$t_1 = t_x \to \text{int}$
	$t_3 = \text{int}$
	$t_4 = \text{int}$
	$t_2 = \text{int}$

Example 1

Consider the next equation $t_f = \text{int} \rightarrow t_3$. The equation contains t_3, which is already bound to int in the substitution. Substitute int for t_3 in the equation. This is called *applying* the substitution to the equation.

Equations	Substitution
$t_f = \text{int} \rightarrow \text{int}$	$t_0 = t_f \rightarrow (t_x \rightarrow \text{int})$
$t_f = t_x \rightarrow t_4$	$t_1 = t_x \rightarrow \text{int}$
	$t_3 = \text{int}$
	$t_4 = \text{int}$
	$t_2 = \text{int}$

Move the resulting equation to the substitution and update it.

Equations	Substitution
$t_f = t_x \rightarrow t_4$	$t_0 = (\text{int} \rightarrow \text{int}) \rightarrow (t_x \rightarrow \text{int})$
	$t_1 = t_x \rightarrow \text{int}$
	$t_3 = \text{int}$
	$t_4 = \text{int}$
	$t_2 = \text{int}$
	$t_f = \text{int} \rightarrow \text{int}$

Example 1

Apply the substitution to the equation:

Equations	Substitution
$\text{int} \to \text{int} \;=\; t_x \to \text{int}$	$t_0 = (\text{int} \to \text{int}) \to (t_x \to \text{int})$
	$t_1 = t_x \to \text{int}$
	$t_3 = \text{int}$
	$t_4 = \text{int}$
	$t_2 = \text{int}$
	$t_f = \text{int} \to \text{int}$

If neither side of the equation is a variable, simplify the equation by yielding two new equations:

Equations	Substitution
$\text{int} = t_x$	$t_0 = (\text{int} \to \text{int}) \to (t_x \to \text{int})$
$\text{int} = \text{int}$	$t_1 = t_x \to \text{int}$
	$t_3 = \text{int}$
	$t_4 = \text{int}$
	$t_2 = \text{int}$
	$t_f = \text{int} \to \text{int}$

Example 1

Switch the sides of the first equation and move it to the substitution:

Equations	Substitution
$\text{int} = \text{int}$	$t_0 = (\text{int} \to \text{int}) \to (\text{int} \to \text{int})$
	$t_1 = \text{int} \to \text{int}$
	$t_3 = \text{int}$
	$t_4 = \text{int}$
	$t_2 = \text{int}$
	$t_f = \text{int} \to \text{int}$
	$t_x = \text{int}$

The final substitution is the solution of the original equations.

Example 2

$$\underbrace{\texttt{proc}\ (\underbrace{f}_{t_f})\ (\underbrace{f\ 11}_{t_1})}_{t_0}$$

$$\begin{array}{rcl} t_0 & = & t_f \to t_1 \\ t_f & = & \mathsf{int} \to t_1 \end{array}$$

Example 2

1.

Equations	Substitution
$t_0 = t_f \to t_1$	
$t_f = \text{int} \to t_1$	

2.

Equations	Substitution
$t_f = \text{int} \to t_1$	$t_0 = t_f \to t_1$

3.

Equations	Substitution
	$t_0 = (\text{int} \to t_1) \to t_1$
	$t_f = \text{int} \to t_1$

The type is *polymorphic* in t_1.

Example 3

$$\underbrace{\texttt{if } \underbrace{x}_{t_x} \texttt{ then } \underbrace{(x-1)}_{t_1} \texttt{ else } 0}_{t_0}$$

$$\begin{array}{rcl} t_x & = & \text{bool} \\ t_1 & = & t_0 \\ \text{int} & = & t_0 \\ t_x & = & \text{int} \\ t_1 & = & \text{int} \end{array}$$

Example 3

The equations have no solutions because, during the unification algorithm, we encounter the following contradictory state:

Equations	Substitution
bool = int	t_x = bool
t_1 = int	t_1 = int
	t_0 = int

Because bool and int cannot be equal, there is no solution to the equations.

Example 4

$$\underbrace{\texttt{proc}\ (\underbrace{f}_{t_f})\ \underbrace{(\texttt{iszero}\ \underbrace{(f\ f)}_{t_2})}_{t_1}}_{t_0}$$

$$\begin{array}{rcl} t_0 & = & t_f \rightarrow t_1 \\ t_1 & = & \text{bool} \\ t_2 & = & \text{int} \\ t_f & = & t_f \rightarrow t_2 \end{array}$$

Example 4

Solving as usual, we encounter a problem:

Equations	Substitution
$t_f = t_f \to \text{int}$	$t_0 = t_f \to \text{bool}$ $t_1 = \text{bool}$ $t_2 = \text{int}$

- There is no type t_f that satisfies the equation, because the right-hand side of the equation is always larger than the left.
- If we ever deduce an equation of the form $t = \ldots t \ldots$ where the type variable t occurs in the right-hand side, we must conclude that there is no solution. This is called *occurrence check*.

Unification Algorithm

For each equation in turn,

- Apply the current substitution to the equation.
- If the equation is always true (e.g. $\text{int} = \text{int}$), discard it.
- If the left- and right-hand sides are contradictory (e.g. $\text{bool} = \text{int}$), the algorithm fails.
- If neither side is a variable (e.g. $\text{int} \to \boldsymbol{t_1} = \boldsymbol{t_2} \to \text{bool}$), simplify the equation, which eventually generates an equation whose left- or right-hand side is a variable.
- If the left-hand side is not a variable, switch the sides.
- If the left-hand side variable occurs in the right-hand side, the algorithm fails.
- Otherwise, move it to the substitution and substitute the right-hand side for each occurrence of the variable in the substitution.

Exercise 1

$$\texttt{let}\ x = 4\ \texttt{in}\ (x\ 3)$$

Exercise 2

$$\texttt{let}\ f\ = \texttt{proc}\ (z)\ z\ \texttt{in}\ \texttt{proc}\ (x)\ ((f\ x) - 1)$$

Exercise 3

$$\texttt{let}\ p\ = \texttt{iszero}\ \mathbf{1}\ \texttt{in}\ \texttt{if}\ p\ \texttt{then}\ \mathbf{88}\ \texttt{else}\ \mathbf{99}$$

Exercise 4

$$\texttt{let}\ f\ = \texttt{proc}\ (x)\ x\ \texttt{in}\ \texttt{if}\ (f\ (\texttt{iszero}\mathbf{0}))\ \texttt{then}\ (f\ \mathbf{11})\ \texttt{else}\ (f\ \mathbf{22})$$

Substitution

Solutions of type equations are represented by substitution:

$$S \in Subst = TyVar \to T$$

Applying a substitution to a type:

$$\begin{array}{rcl}
S(\text{int}) & = & \text{int} \\
S(\text{bool}) & = & \text{bool} \\
S(\alpha) & = & \begin{cases} t & \text{if } \alpha \mapsto t \in S \\ \alpha & \text{otherwise} \end{cases} \\
S(T_1 \to T_2) & = & S(T_1) \to S(T_2)
\end{array}$$

Example

Applying the substitution

$$S = \{t_1 \mapsto \mathsf{int}, t_2 \mapsto \mathsf{int} \rightarrow \mathsf{int}\}$$

to to the type $(t_1 \rightarrow t_2) \rightarrow (t_3 \rightarrow \mathsf{int})$:

$$\begin{aligned}
& S((t_1 \rightarrow t_2) \rightarrow (t_3 \rightarrow \mathsf{int})) \\
& = S(t_1 \rightarrow t_2) \rightarrow S(t_3 \rightarrow \mathsf{int}) \\
& = (S(t_1) \rightarrow S(t_2)) \rightarrow (S(t_3) \rightarrow S(\mathsf{int})) \\
& = (\mathsf{int} \rightarrow (\mathsf{int} \rightarrow \mathsf{int})) \rightarrow (t_3 \rightarrow \mathsf{int})
\end{aligned}$$

Unification

Update the current substitution with equality $t_1 \doteq t_2$.

$$\textbf{unify} : T \times T \times Subst \to Subst$$

$$
\begin{array}{rcl}
\textbf{unify}(\text{int}, \text{int}, S) & = & S \\
\textbf{unify}(\text{bool}, \text{bool}, S) & = & S \\
\textbf{unify}(\alpha, \alpha, S) & = & S \\
\textbf{unify}(\alpha, t, S) & = & \begin{cases} \text{fail} & \alpha \text{ occurs in } t \\ \text{extend } S \text{ with } \alpha \doteq t & \text{otherwise} \end{cases} \\
\textbf{unify}(t, \alpha, S) & = & \textbf{unify}(\alpha, t, S) \\
\textbf{unify}(t_1 \to t_2, t_1' \to t_2', S) & = & \text{let } S' = \textbf{unify}(t_1, t_1', S) \text{ in} \\
 & & \text{let } S'' = \textbf{unify}(S'(t_2), S'(t_2'), S') \text{ in} \\
 & & \quad S'' \\
\textbf{unify}(\_, \_, \_) & = & \text{fail}
\end{array}
$$

Exercises

- $\textbf{unify}(\alpha, \text{int} \to \text{int}, \emptyset) =$
- $\textbf{unify}(\alpha, \text{int} \to \alpha, \emptyset) =$
- $\textbf{unify}(\alpha \to \beta, \text{int} \to \text{int}, \emptyset) =$
- $\textbf{unify}(\alpha \to \beta, \text{int} \to \alpha, \emptyset) =$
- $\textbf{unify}(\alpha \to \beta, \beta \to \text{int}, \emptyset) =$

Solving Equations

$$\textbf{unifyall} : TyEqn \rightarrow Subst \rightarrow Subst$$

$$\begin{array}{rcl} \textbf{unifyall}(\emptyset, S) & = & S \\ \textbf{unifyall}((t_1 \doteq t_2) \ \wedge \ u, S) & = & \text{let } S' = \textbf{unify}(S(t_1), S(t_2), S) \\ & & \text{in } \textbf{unifyall}(u, S') \end{array}$$

Let $\mathcal{U}$ be the final unification algorithm:

$$\mathcal{U}(u) = \textbf{unifyall}(u, \emptyset)$$

typeof : $E \to T$

The final type inference algorithm that composes equation derivation ($\boldsymbol{\mathcal{V}}$) and equation solving ($\boldsymbol{\mathcal{U}}$):

$$
\begin{array}{l}
\textbf{typeof}(E) = \\
\quad \textbf{let } S = \mathcal{U}(\mathcal{V}(\emptyset, E, \alpha)) \quad (\text{new } \alpha) \\
\quad \textbf{in } S(\alpha)
\end{array}
$$

Examples

- $\textbf{typeof}((\texttt{proc}\ (x)\ x)\ 1)$
- $\textbf{typeof}(\texttt{let}\ x = 1\ \texttt{in}\ \texttt{proc}(y)\ (x + y))$

Summary

Automatic type inference:

- derive type equations from the program text, and
- solve the equations by unification algorithm.