COSE212: Programming Languages

Lecture 18 — Semantics of Programming Languages (Operational and Denotational Semantics)

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2023 Fall

The **While** Language: Abstract Syntax

n will range over numerals, **Num**
x will range over variables, **Var**
a will range over arithmetic expressions, **Aexp**
b will range over boolean expressions, **Bexp**
c, S will range over statements, **Stm**

$$
\begin{array}{rcl}
a & \rightarrow & n \mid x \mid a_1 + a_2 \mid a_1 \star a_2 \mid a_1 - a_2 \\
b & \rightarrow & \texttt{true} \mid \texttt{false} \mid a_1 = a_2 \mid a_1 \leq a_2 \mid \neg b \mid b_1 \wedge b_2 \\
c & \rightarrow & x := a \mid \texttt{skip} \mid c_1; c_2 \mid \texttt{if}\ b\ c_1\ c_2 \mid \texttt{while}\ b\ c
\end{array}
$$

Example

The factorial program:

```
y:=1; while ¬(x=1) do (y:=y⋆x; x:=x-1)
```

The abstract syntax tree:

Plan

- Operational semantics
  - Big-step operational semantics for **While**
  - Small-step operational semantics for **While**
  - Implementing Interpreters
- Denotational semantics

Syntax vs. Semantics

A programming language is defined with syntax and semantics.

- The syntax is concerned with the grammatical structure of programs.
  - Context-free grammar
- The semantics is concerned with the meaning of grammatically correct programs.
  - Operational semantics: The meaning is specified by the computation steps executed on a machine. It is of intrest how it is obtained.
  - Denotational semantics: The meaning is modelled by mathematical objects that represent the effect of executing the program. It is of interest the effect, not how it is obtained.

Semantics of Arithmetic Expressions

- The meaning of an expression depends on the values bound to the variables that occur in the expression, e.g., $x + 3$.
- A state is a function from variables to values:

$$\textsf{State} = \mathbf{Var} \rightarrow \mathbb{Z}$$

- The meaning of arithmetic expressions is a function:

$$\mathcal{A} : \mathbf{Aexp} \rightarrow \textsf{State} \rightarrow \mathbb{Z}$$

$$\begin{array}{rcl}
\mathcal{A}[\![a]\!] & : & \textsf{State} \rightarrow \mathbb{Z} \\
\mathcal{A}[\![n]\!](s) & = & n \\
\mathcal{A}[\![x]\!](s) & = & s(x) \\
\mathcal{A}[\![a_1 + a_2]\!](s) & = & \mathcal{A}[\![a_1]\!](s) + \mathcal{A}[\![a_2]\!](s) \\
\mathcal{A}[\![a_1 \star a_2]\!](s) & = & \mathcal{A}[\![a_1]\!](s) \times \mathcal{A}[\![a_2]\!](s) \\
\mathcal{A}[\![a_1 - a_2]\!](s) & = & \mathcal{A}[\![a_1]\!](s) - \mathcal{A}[\![a_2]\!](s)
\end{array}$$

Semantics of Boolean Expressions

- The meaning of boolean expressions is a function:

$$\mathcal{B} : \mathrm{Bexp} \to \textsf{State} \to \textsf{T}$$

where $\textsf{T} = \{true, false\}$.

$$\begin{array}{rcl}
\mathcal{B}[\![b]\!] & : & \textsf{State} \to \textsf{T} \\
\mathcal{B}[\![\texttt{true}]\!](s) & = & true \\
\mathcal{B}[\![\texttt{false}]\!](s) & = & false \\
\mathcal{B}[\![a_1 = a_2]\!](s) & = & \mathcal{A}[\![a_1]\!](s) = \mathcal{A}[\![a_2]\!](s) \\
\mathcal{B}[\![a_1 \leq a_2]\!](s) & = & \mathcal{A}[\![a_1]\!](s) \leq \mathcal{A}[\![a_2]\!](s) \\
\mathcal{B}[\![\neg b]\!](s) & = & \mathcal{B}[\![b]\!](s) = false \\
\mathcal{B}[\![b_1 \wedge b_2]\!](s) & = & \mathcal{B}[\![b_1]\!](s) \wedge \mathcal{B}[\![b_2]\!](s)
\end{array}$$

Free Variables

The free variables of an arithmetic expression a are defined to be the set of variables occurring in it:

$$
\begin{array}{rcl}
FV(n) & = & \emptyset \\
FV(x) & = & \{x\} \\
FV(a_1 + a_2) & = & FV(a_1) \cup FV(a_2) \\
FV(a_1 \star a_2) & = & FV(a_1) \cup FV(a_2) \\
FV(a_1 - a_2) & = & FV(a_1) \cup FV(a_2)
\end{array}
$$

Exercise) Define free variables of boolean expressions.

Substitution

- $a[y \mapsto a_0]$: the arithmetic expression that is obtained by replacing each occurrence of y in a by a_0.

$$\begin{array}{rcl} n[y \mapsto a_0] & = & n \\ x[y \mapsto a_0] & = & \begin{cases} a_0 & \text{if } x = y \\ x & \text{if } x \neq y \end{cases} \\ (a_1 + a_2)[y \mapsto a_0] & = & (a_1[y \mapsto a_0]) + (a_2[y \mapsto a_0]) \\ (a_1 \star a_2)[y \mapsto a_0] & = & (a_1[y \mapsto a_0]) \star (a_2[y \mapsto a_0]) \\ (a_1 - a_2)[y \mapsto a_0] & = & (a_1[y \mapsto a_0]) - (a_2[y \mapsto a_0]) \end{array}$$

- $s[y \mapsto v]$: the state s except that the value bound to y is v.

$$(s[y \mapsto v])(x) = \begin{cases} v & \text{if } x = y \\ s(x) & \text{if } x \neq y \end{cases}$$

Lemma (The two concepts of substitutions are related)

$\mathcal{A}[\![a[y \mapsto a_0]]\!](s) = \mathcal{A}[\![a]\!](s[y \mapsto \mathcal{A}[\![a_0]\!](s)])$ *for all states* s.

Operational Semantics

Operational semantics is concerned about how to execute programs and not merely what the execution results are.

- *Big-step operational semantics* describes how the overall results of executions are obtained.
- *Small-step operational semantics* describes how the individual steps of the computations take place.

In both kinds, the semantics is specified by a transition system $(\mathbb{S}, \rightarrow)$ where $\mathbb{S}$ is the set of states (configurations) with two types:

- $\langle S, s \rangle$: a nonterminal state (i.e. the statement S is to be executed from the state s)
- s: a terminal state

The transition relation $(\rightarrow) \subseteq \mathbb{S} \times \mathbb{S}$ describes how the execution takes place. The difference between the two approaches are in the definitions of transition relation.

Big-step Operational Semantics

The transition relation specifies the relationship between the initial state and the final state:

$$\langle S, s\rangle \to s'$$

Transition relation is defined with inference rules of the form: A rule has the general form

$$\frac{\langle S_1, s_1\rangle \to s'_1, \ldots, \langle S_n, s_n\rangle \to s'_n}{\langle S, s\rangle \to s'} \text{ if } \cdots$$

- $S_1, \ldots, S_n$ are statements that constitute S.
- A rule has a number of premises and one conclusion.
- A rule may also have a number of conditions that have to be fulfilled whenever the rule is applied.
- Rules without premises are called axioms.

Big-step Operational Semantics for **While**

$$\frac{}{\langle x := a, s\rangle \to s[x \mapsto \mathcal{A}[\![a]\!](s)]}$$

$$\frac{}{\langle \texttt{skip}, s\rangle \to s}$$

$$\frac{\langle S_1, s\rangle \to s' \qquad \langle S_2, s'\rangle \to s''}{\langle S_1; S_2, s\rangle \to s''}$$

$$\frac{\langle S_1, s\rangle \to s'}{\langle \texttt{if}\ b\ S_1\ S_2, s\rangle \to s'} \text{ if } \mathcal{B}[\![b]\!](s) = \textbf{true}$$

$$\frac{\langle S_2, s\rangle \to s'}{\langle \texttt{if}\ b\ S_1\ S_2, s\rangle \to s'} \text{ if } \mathcal{B}[\![b]\!](s) = \textbf{false}$$

$$\frac{\langle S, s\rangle \to s' \qquad \langle \texttt{while}\ b\ S, s'\rangle \to s''}{\langle \texttt{while}\ b\ S, s\rangle \to s''} \text{ if } \mathcal{B}[\![b]\!](s) = \textbf{true}$$

$$\frac{}{\langle \texttt{while}\ b\ S, s\rangle \to s} \text{ if } \mathcal{B}[\![b]\!](s) = \textbf{false}$$

Example

Consider the statement:

```
(z:=x; x:=y); y:=z
```

Let s_0 be the state that maps all variables except x and y and has $s_0(\mathrm{x}) = 5$ and $s_0(\mathrm{y}) = 7$.

$$\dfrac{\dfrac{\langle \mathrm{z} := \mathrm{x}, s_0 \rangle \to s_1 \qquad \langle \mathrm{x} := \mathrm{y}, s_1 \rangle \to s_2}{\langle \mathrm{z} := \mathrm{x}; \mathrm{x} := \mathrm{y}, s_0 \rangle \to s_2} \qquad \langle \mathrm{y} := \mathrm{z}, s_2 \rangle \to s_3}{\langle (\mathrm{z} := \mathrm{x}; \mathrm{x} := \mathrm{y}); \mathrm{y} := \mathrm{z}, s_0 \rangle \to s_3}$$

where we have used the abbreviations:

$$\begin{array}{rcl} s_1 & = & s_0[z \mapsto 5] \\ s_2 & = & s_1[x \mapsto 7] \\ s_3 & = & s_2[y \mapsto 5] \end{array}$$

Exercise

Let s be a state with $s(x) = 3$. Find s' such that

$$(\texttt{y:=1; while } \neg\texttt{(x=1) do (y:=y}\star\texttt{x; x:=x-1)}, s) \rightarrow s'$$

Execution Types

We say the execution of a statement S on a state s

- *terminates* if and only if there is a state s' such that $\langle S, s\rangle \to s'$ and
- *loops* if and only if there is no state s' such that $\langle S, s\rangle \to s'$.

We say a statement S always terminates if its execution on a state s terminates for all states s, and always loops if its execution on a state s loops for all states s.

Examples:

- `while true do skip`
- `while ¬(x=1) do (y:=y⋆x; x:=x-1)`

Semantic Equivalence

We say S_1 and S_2 are semantically equivalent, denoted $S_1 \equiv S_2$, if the following is true for all states s and s':

$$\langle S_1, s\rangle \to s' \quad \text{if and only if} \quad \langle S_2, s\rangle \to s'$$

Example:

$$\texttt{while}\ b\ \texttt{do}\ S \equiv \texttt{if}\ b\ \texttt{then}\ (S;\ \texttt{while}\ b\ \texttt{do}\ S)\ \texttt{else}\ \texttt{skip}$$

Semantic Function for Statements

The semantic function for statements is the partial function:

$$\mathcal{S}_b : \mathrm{Stm} \to (\textbf{State} \hookrightarrow \textbf{State})$$

$$\mathcal{S}_b[\![S]\!](s) = \begin{cases} s' & \text{if } \langle S, s\rangle \to s' \\ \textbf{undef} & \text{otherwise} \end{cases}$$

Examples:

- $\mathcal{S}_b[\![\texttt{y:=1; while } \neg\texttt{(x=1) do (y:=y}\star\texttt{x; x:=x-1)}]\!](s[x \mapsto 3])$
- $\mathcal{S}_b[\![\texttt{while true do skip}]\!](\texttt{s})$

Summary of Big-step Semantics

The syntax is defined by the grammar:

$$\begin{array}{rcl} a & \rightarrow & n \mid x \mid a_1 + a_2 \mid a_1 \star a_2 \mid a_1 - a_2 \\ b & \rightarrow & \texttt{true} \mid \texttt{false} \mid a_1 = a_2 \mid a_1 \leq a_2 \mid \neg b \mid b_1 \wedge b_2 \\ c & \rightarrow & x := a \mid \texttt{skip} \mid c_1; c_2 \mid \texttt{if}\ b\ c_1\ c_2 \mid \texttt{while}\ b\ c \end{array}$$

The semantics is defined by the functions:

$$\begin{array}{rcl} \mathcal{A}[\![a]\!] & : & \textbf{State} \rightarrow \mathbb{Z} \\ \mathcal{B}[\![b]\!] & : & \textbf{State} \rightarrow \textbf{T} \\ \mathcal{S}_b[\![c]\!] & : & \textbf{State} \hookrightarrow \textbf{State} \end{array}$$

Implementing Big-step Interpreter in OCaml

```
type var = string

type aexp =
  | Int of int
  | Var of var
  | Plus of aexp * aexp
  | Mult of aexp * aexp
  | Minus of aexp * aexp

type bexp =
  | True
  | False
  | Eq of aexp * aexp
  | Le of aexp * aexp
  | Neg of bexp
  | Conj of bexp * bexp

type cmd =
  | Assign of var * aexp
  | Skip
  | Seq of cmd * cmd
  | If of bexp * cmd * cmd
  | While of bexp * cmd
```

Implementing Big-step Interpreter

```
let fact =
  Seq (Assign ("y", Int 1),
    While (Neg (Eq (Var "x", Int 1)),
      Seq (Assign("y", Mult(Var "y", Var "x")),
           Assign("x", Minus(Var "x", Int 1)))
       )
  )

module State = struct
  type t = (var * int) list
  let empty = []
  let rec lookup s x =
    match s with
    | [] -> raise (Failure (x ^ "is not bound in state"))
    | (y,v)::s' -> if x = y then v else lookup s' x
  let update s x v = (x,v)::s
end

let init_s = update empty "x" 3
```

Implementing Big-step Interpreter

```
let rec eval_a : aexp -> State.t -> int
=fun a s ->
  match a with
  | Int n -> n
  | Var x -> State.lookup s x
  | Plus (a1, a2) -> (eval_a a1 s) + (eval_a a2 s)
  | Mult (a1, a2) -> (eval_a a1 s) * (eval_a a2 s)
  | Minus (a1, a2) -> (eval_a a1 s) - (eval_a a2 s)

let rec eval_b : bexp -> State.t -> bool
=fun b s ->
  match b with
  | True -> true
  | False -> false
  | Eq (a1, a2) -> (eval_a a1 s) = (eval_a a2 s)
  | Le (a1, a2) -> (eval_a a1 s) <= (eval_a a2 s)
  | Neg b' -> not (eval_b b' s)
  | Conj (b1, b2) -> (eval_b b1 s) && (eval_b b2 s)
```

Implementing Big-Step Interpreter

```
let rec eval_c : cmd -> State.t -> State.t
=fun c s ->
  match c with
  | Assign (x, a) -> State.update s x (eval_a a s)
  | Skip -> s
  | Seq (c1, c2) -> eval_c c2 (eval_c c1 s)
  | If (b, c1, c2) -> eval_c (if eval_b b s then c1 else c2) s
  | While (b, c) ->
    if eval_b b s then eval_c (While (b,c)) (eval_c c s)
    else s

let _ =
    print_int (State.lookup (eval_c fact init_s) "y");
    print_newline ()
```

Small-step Operational Semantics

The individual computation steps are described by the transition relation of the form:

$$\langle S, s \rangle \Rightarrow \gamma$$

where γ either is non-terminal state $\langle S', s' \rangle$ or terminal state s'. The transition expresses the first step of the execution of S from state s.

- If $\gamma = \langle S', s' \rangle$, then the execution of S from s is not completed and the remaining computation continues with $\langle S', s' \rangle$.
- If $\gamma = s'$, then the execution of S from s has terminated and the final state is s'.

We say $\langle S, s \rangle$ is stuck if there is no γ such that $\langle S, s \rangle \Rightarrow \gamma$ (no stuck state for **While**).

Small-step Operational Semantics for **While**

$$\frac{}{\langle x := a, s\rangle \Rightarrow s[x \mapsto \mathcal{A}[\![a]\!](s)]}$$

$$\frac{}{\langle \texttt{skip}, s\rangle \Rightarrow s}$$

$$\frac{\langle S_1, s\rangle \Rightarrow \langle S_1', s'\rangle}{\langle S_1; S_2, s\rangle \Rightarrow \langle S_1'; S_2, s'\rangle}$$

$$\frac{\langle S_1, s\rangle \Rightarrow s'}{\langle S_1; S_2, s\rangle \Rightarrow \langle S_2, s'\rangle}$$

$$\frac{}{\langle \texttt{if}\ b\ S_1\ \ S_2, s\rangle \Rightarrow \langle S_1, s\rangle}\ \text{if } \mathcal{B}[\![b]\!](s) = \textbf{true}$$

$$\frac{}{\langle \texttt{if}\ b\ S_1\ \ S_2, s\rangle \Rightarrow \langle S_2, s\rangle}\ \text{if } \mathcal{B}[\![b]\!](s) = \textbf{false}$$

$$\frac{}{\langle \texttt{while}\ b\ S, s\rangle \Rightarrow \langle \texttt{if}\ \ b\ \ \texttt{(}S\texttt{;}\ \ \texttt{while}\ \ b\ \ S\texttt{)}\ \ \texttt{skip}, s\rangle}$$

Derivation Sequence

A *derivation sequence* of a statement S starting in state s is either

- A finite sequence

  $$\gamma_0, \gamma_1, \gamma_2, \cdots, \gamma_k$$

  which is sometimes written

  $$\gamma_0 \Rightarrow \gamma_1 \Rightarrow \gamma_2 \Rightarrow \cdots \Rightarrow \gamma_k$$

  such that

  $$\gamma_0 = \langle S, s \rangle, \quad \gamma_i \Rightarrow \gamma_{i+1} \text{ for } 0 \leq i \leq k$$

  and γ_k is either a terminal configuration or a stuck configuration.
- An infinite sequence

  $$\gamma_0, \gamma_1, \gamma_2, \cdots$$

  which is sometimes written

  $$\gamma_0 \Rightarrow \gamma_1 \Rightarrow \gamma_2 \Rightarrow \cdots$$

  consisting of configurations satisfying $\gamma_0 = \langle S, s \rangle$ and $\gamma_i \Rightarrow \gamma_{i+1}$ for $0 \leq i$.

Example

Consider the statement:

```
(z:=x; x:=y); y:=z
```

Let s_0 be the state that maps all variables except x and y and has $s_0(\mathrm{x}) = 5$ and $s_0(\mathrm{y}) = 7$. We then have the derivation sequence:

$$\begin{array}{l} \langle (\mathrm{z} := \mathrm{x}; \mathrm{x} := \mathrm{y}); \mathrm{y} := \mathrm{z}, s_0 \rangle \\ \Rightarrow \langle \mathrm{x} := \mathrm{y}; \mathrm{y} := \mathrm{z}, s_0[z \mapsto 5] \rangle \\ \Rightarrow \langle \mathrm{y} := \mathrm{z}, s_0[z \mapsto 5, x \mapsto 7] \rangle \\ \Rightarrow s_0[z \mapsto 5, x \mapsto 7, y \mapsto 5] \end{array}$$

Each step has a derivation tree explaining why it takes place, e.g.,

$$\frac{\dfrac{\langle \mathrm{z} := \mathrm{x}, s_0 \rangle \Rightarrow s_0[z \mapsto 5]}{\langle \mathrm{z} := \mathrm{x}; \mathrm{x} := \mathrm{y}, s_0 \rangle \Rightarrow \langle \mathrm{x} := \mathrm{y}, s_0[z \mapsto 5] \rangle}}{\langle (\mathrm{z} := \mathrm{x}; \mathrm{x} := \mathrm{y}); \mathrm{y} := \mathrm{z}, s_0 \rangle \Rightarrow \langle \mathrm{x} := \mathrm{y}; \mathrm{y} := \mathrm{z}, s_0[z \mapsto 5] \rangle}$$

Example: Factorial

Assume that $s(x) = 3$.

$\langle$`y:=1; while ¬(x=1) do (y:=y⋆x; x:=x-1)`$, s\rangle$
$\Rightarrow \langle$`while ¬(x=1) do (y:=y⋆x; x:=x-1)`$, s[y \mapsto 1]\rangle$
$\Rightarrow \langle$`if ¬(x=1) then ((y:=y⋆x; x:=x-1);while ¬(x=1) do (y:=y⋆x; x:=x-1)) else skip`$, s[y \mapsto 1]\rangle$
$\Rightarrow \langle$`(y:=y⋆x; x:=x-1);while ¬(x=1) do (y:=y⋆x; x:=x-1)`$, s[y \mapsto 1]\rangle$
$\Rightarrow \langle$`x:=x-1;while ¬(x=1) do (y:=y⋆x; x:=x-1)`$, s[y \mapsto 3]\rangle$
$\Rightarrow \langle$`while ¬(x=1) do (y:=y⋆x; x:=x-1)`$, s[y \mapsto 3][x \mapsto 2]\rangle$
$\Rightarrow \langle$`if ¬(x=1) then ((y:=y⋆x; x:=x-1);while ¬(x=1) do (y:=y⋆x; x:=x-1)) else skip`$, s[y \mapsto 3][x \mapsto 2]\rangle$
$\Rightarrow \langle$`(y:=y⋆x; x:=x-1);while ¬(x=1) do (y:=y⋆x; x:=x-1)`$, s[y \mapsto 3][x \mapsto 2]\rangle$
$\Rightarrow \langle$`x:=x-1;while ¬(x=1) do (y:=y⋆x; x:=x-1)`$, s[y \mapsto 6][x \mapsto 2]\rangle$
$\Rightarrow \langle$`while ¬(x=1) do (y:=y⋆x; x:=x-1)`$, s[y \mapsto 6][x \mapsto 1]\rangle$
$\Rightarrow^* s[y \mapsto 6][x \mapsto 1]$

Other Notations

- We write $\gamma_0 \Rightarrow^i \gamma_i$ to indicate that there are i steps in the execution from γ_0 to γ_i.
- We write $\gamma_0 \Rightarrow^* \gamma_i$ to indicate that there are a finite number of steps.
- We say that the execution of a statement S on a state s terminates if and only if there is a finite derivation sequence starting with $\langle S, s \rangle$.
- The execution loops if and only if there is an infinite derivation sequence starting with $\langle S, s \rangle$.

Semantic Equivalence

We say S_1 and S_2 are semantically equivalent if for all states s,

- $\langle S_1, s\rangle \Rightarrow^* \gamma$ if and only if $\langle S_2, s\rangle \Rightarrow^* \gamma$, whenever γ is a configuration that is either stuck or terminal, and
- there is an infinite derivation sequence starting in $\langle S_1, s\rangle$ if and only if there is one starting in $\langle S_2, s\rangle$.

Semantic Function

The semantic function $\mathcal{S}_s$ for small-step semantics:

$$\mathcal{S}_s : \mathrm{Stm} \to (\textbf{State} \hookrightarrow \textbf{State})$$

$$\mathcal{S}_s[\![S]\!](s) = \begin{cases} s' & \text{if } \langle S, s\rangle \Rightarrow^* s' \\ \textbf{undef} & \end{cases}$$

Implementing Small-Step Interpreter

```
type conf =
  | NonTerminated of cmd * State.t
  | Terminated of State.t

let rec next : conf -> conf
=fun conf ->
  match conf with
  | Terminated _ -> raise (Failure "Must not happen")
  | NonTerminated (c, s) ->
    match c with
    | Assign (x, a) -> Terminated (State.update s x (eval_a a s))
    | Skip -> Terminated s
    | Seq (c1, c2) -> (
       match  (next (NonTerminated (c1,s))) with
       | NonTerminated (c', s') -> NonTerminated (Seq (c', c2), s')
       | Terminated s' -> NonTerminated (c2, s')
      )
    | If (b, c1, c2) ->
      if eval_b b s then NonTerminated (c1, s) else NonTerminated (c2, s)
    | While (b, c) -> NonTerminated (If (b, Seq (c, While (b,c)), Skip), s)
```

Implementing Small-Step Interpreter

```
let rec next_trans : conf -> State.t
=fun conf ->
  match conf with
  | Terminated s -> s
  | _ -> next_trans (next conf)

let _ =
  print_int (State.lookup (next_trans (NonTerminated (fact,init_s))) "y");
  print_newline ()
```

Summary of Operational Semantics

We have defined the operational semantics of **While**.

- *Big-step operational semantics* describes how the overall results of executions are obtained.
- *Small-step operational semantics* describes how the individual steps of the computations take place.

The big-step and small-step operational semantics are equivalent:

Theorem

For every statement S *of* **While**, *we have* $\mathcal{S}_b[\![S]\!] = \mathcal{S}_s[\![S]\!]$.

Denotational Semantics

- In denotational semantics, we are interested in the mathematical meaning of a program.
- Also called compositional semantics: The meaning of an expression is defined with the meanings of its immediate subexpressions.
- Denotational semantics for **While**:

$$
\begin{array}{rcl}
a & \to & n \mid x \mid a_1 + a_2 \mid a_1 \star a_2 \mid a_1 - a_2 \\
b & \to & \texttt{true} \mid \texttt{false} \mid a_1 = a_2 \mid a_1 \leq a_2 \mid \neg b \mid b_1 \land b_2 \\
c & \to & x := a \mid \texttt{skip} \mid c_1; c_2 \mid \texttt{if}\ b\ c_1\ c_2 \mid \texttt{while}\ b\ c
\end{array}
$$

Denotational Semantics of Expressions

$$\begin{array}{rcl}
\mathcal{A}[\![a]\!] & : & \textsf{State} \to \mathbb{Z} \\
\mathcal{A}[\![n]\!](s) & = & n \\
\mathcal{A}[\![x]\!](s) & = & s(x) \\
\mathcal{A}[\![a_1 + a_2]\!](s) & = & \mathcal{A}[\![a_1]\!](s) + \mathcal{A}[\![a_2]\!](s) \\
\mathcal{A}[\![a_1 \star a_2]\!](s) & = & \mathcal{A}[\![a_1]\!](s) \times \mathcal{A}[\![a_2]\!](s) \\
\mathcal{A}[\![a_1 - a_2]\!](s) & = & \mathcal{A}[\![a_1]\!](s) - \mathcal{A}[\![a_2]\!](s)
\end{array}$$

$$\begin{array}{rcl}
\mathcal{B}[\![b]\!] & : & \textsf{State} \to \textsf{T} \\
\mathcal{B}[\![\texttt{true}]\!](s) & = & \mathit{true} \\
\mathcal{B}[\![\texttt{false}]\!](s) & = & \mathit{false} \\
\mathcal{B}[\![a_1 = a_2]\!](s) & = & \mathcal{A}[\![a_1]\!](s) = \mathcal{A}[\![a_2]\!](s) \\
\mathcal{B}[\![a_1 \le a_2]\!](s) & = & \mathcal{A}[\![a_1]\!](s) \le \mathcal{A}[\![a_2]\!](s) \\
\mathcal{B}[\![\neg b]\!](s) & = & \mathcal{B}[\![b]\!](s) = \mathit{false} \\
\mathcal{B}[\![b_1 \wedge b_2]\!](s) & = & \mathcal{B}[\![b_1]\!](s) \wedge \mathcal{B}[\![b_2]\!](s)
\end{array}$$

Denotational Semantics of Commands

$$
\begin{array}{rcl}
\mathcal{C}[\![c]\!] & : & \textbf{State} \hookrightarrow \textbf{State} \\
\mathcal{C}[\![x := a]\!](s) & = & s[x \mapsto \mathcal{A}[\![a]\!](s)] \\
\mathcal{C}[\![\texttt{skip}]\!] & = & \textbf{id} \\
\mathcal{C}[\![c_1; c_2]\!] & = & \mathcal{C}[\![c_2]\!] \circ \mathcal{C}[\![c_1]\!] \\
\mathcal{C}[\![\texttt{if}\ b\ c_1\ c_2]\!] & = & \textbf{cond}(\mathcal{B}[\![b]\!], \mathcal{C}[\![c_1]\!], \mathcal{C}[\![c_2]\!]) \\
\mathcal{C}[\![\texttt{while}\ b\ c]\!] & = & \textbf{fix}\ F
\end{array}
$$

where

$$
\textbf{cond}(f, g, h) = \lambda s. \begin{cases} g(s) & \cdots f(s) = true \\ h(s) & \cdots f(s) = false \end{cases}
$$

$$
F(g) = \textbf{cond}(\mathcal{B}[\![b]\!], g \circ \mathcal{C}[\![c]\!], \textbf{id})
$$

Denotational Semantics of Loops

The meaning of the while loop is the mathematical object (i.e. partial function in **State** $\hookrightarrow$ **State**) that satisfies the equation:

$$\mathcal{C}[\![\texttt{while } b\ c]\!] = \textbf{cond}(\mathcal{B}[\![b]\!], \mathcal{C}[\![\texttt{while } b\ c]\!] \circ \mathcal{C}[\![c]\!], \textbf{id}).$$

Rewrite the equation:

$$\mathcal{C}[\![\texttt{while } b\ c]\!] = F(\mathcal{C}[\![\texttt{while } b\ c]\!])$$

where

$$F(g) = \textbf{cond}(\mathcal{B}[\![b]\!], g \circ \mathcal{C}[\![c]\!], \textbf{id}).$$

The meaning of the while loop is defined as the least fixed point of F:

$$\mathcal{C}[\![\texttt{while } b\ c]\!] = \textbf{fix } F$$

where **fix** F denotes the *least fixed point* of F.

Example

$$\texttt{while}\ \neg(x = 0)\ \texttt{skip}$$

- F
- **fix** F

Questions

- Does the least fixed point **fix** F exist?
- Is **fix** F unique?
- How to compute **fix** F?

Fixed Point Theory

Theorem

Let $f : D \to D$ be a continuous function on a CPO D. Then f has a (unique) least fixed point, **fix** (f), *and*

$$\textbf{fix}\ (f) = \bigsqcup_{n \geq 0} f^n(\bot).$$

The denotational semantics is well-defined if

- **State** $\hookrightarrow$ **State** is a CPO, and
- F : (**State** $\hookrightarrow$ **State**) $\to$ (**State** $\hookrightarrow$ **State**) is a continuous function.

Summary

- Operational semantics
  - Big-step operational semantics for **While**
  - Small-step operational semantics for **While**
- Denotational semantics