

# COSE212: Programming Languages

## Lecture 17 — Lambda Calculus (Origin of Programming Languages)

Hakjoo Oh  
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# A Fundamental Question

Programming languages look very different.

- C, C++, Java, OCaml, Haskell, Scala, JavaScript, etc

## Example: QuickSort in C

```
void swap(int* a, int* b) { int t = *a; *a = *b; *b = t; }

int partition (int arr[], int low, int high) {
    int pivot = arr[high];
    int i = (low - 1);

    for (int j = low; j <= high- 1; j++) {
        if (arr[j] <= pivot) {
            i++;
            swap(&arr[i], &arr[j]);
        }
    }
    swap(&arr[i + 1], &arr[high]);
    return (i + 1);
}

void quickSort(int arr[], int low, int high) {
    if (low < high) {
        int pi = partition(arr, low, high);
        quickSort(arr, low, pi - 1);
        quickSort(arr, pi + 1, high);
    }
}
```

## Example: QuickSort in Haskell

```
quicksort [] = []
quicksort (x:xs) = quicksort ys ++ [x] ++ quicksort zs
  where
    ys = [a | a <- xs, a <=x]
    zs = [b | b <- xs, b > x]
```

# A Fundamental Question

Are they different fundamentally? or Is there a core mechanism underlying all programming languages?

# Syntactic Sugar

- Syntactic sugar is syntax that makes a language “sweet”: it does not add expressiveness but makes programs easier to read and write.
- For example, we can “desugar” the `let` expression:

$$\text{let } x = E_1 \text{ in } E_2 \xrightarrow{\text{desugar}} (\text{proc } x E_2) E_1$$

- Exercise) Desugar the program:

```
let x = 1 in
  let y = 2 in
    x + y
```

# Syntactic Sugar

Q) Identify all syntactic sugars of the language:

$$\begin{array}{l} E \rightarrow n \\ | \\ | x \\ | \\ | E + E \\ | \\ | E - E \\ | \\ | \text{iszero } E \\ | \\ | \text{if } E \text{ then } E \text{ else } E \\ | \\ | \text{let } x = E \text{ in } E \\ | \\ | \text{letrec } f(x) = E \text{ in } E \\ | \\ | \text{proc } x E \\ | \\ | E E \end{array}$$

# Lambda Calculus ( $\lambda$ -Calculus)

- By removing all syntactic sugars from the language, we obtain a minimal language, called *lambda calculus*:

$$\begin{array}{lll} e & \rightarrow & x \quad \text{variables} \\ & | & \lambda x.e \quad \text{abstraction} \\ & | & e e \quad \text{application} \end{array}$$

Programming language = Lambda calculus + Syntactic sugars



# Origins of Programming Languages and Computer

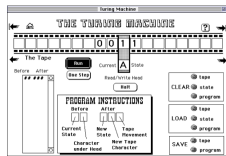


- In 1935, Church developed  $\lambda$ -calculus as a formal system for mathematical logic and argued that any computable function on natural numbers can be computed with  $\lambda$ -calculus. Since then,  $\lambda$ -calculus became the model of programming languages.
- In 1936, Turing independently developed Turing machine and argued that any computable function on natural numbers can be computed with the machine. Since then, Turing machine became the model of computers.

# Church-Turing Thesis

- A surprising fact is that the classes of  $\lambda$ -calculus and Turing machines can compute coincide even though they were developed independently.
- Church and Turing proved that the classes of computable functions defined by  $\lambda$ -calculus and Turing machine are equivalent.

$$\begin{array}{ccc} e & \rightarrow & x \\ | & & \lambda x.e \\ | & & e e \end{array} =$$



A function is  $\lambda$ -computable if and only if Turing computable.

- This equivalence has led mathematicians and computer scientists to believe that these models are “universal”: A function is computable if and only if  $\lambda$ -computable if and only if Turing computable.

# $\lambda$ -Calculus is Everywhere

$\lambda$ -calculus had immense impacts on programming languages.

- It has been the core of functional programming languages (e.g., Lisp, ML, Haskell, Scala, etc).
- Lambdas in other languages:

- ▶ Java8

```
(int n, int m) -> n + m
```

- ▶ C++11

```
[](int x, int y) { return x + y; }
```

- ▶ Python

```
(lambda x, y: x + y)
```

- ▶ JavaScript

```
function (a, b) { return a + b }
```

# Syntax of Lambda Calculus

$e$	$\rightarrow$	$x$	variables
		$\lambda x.e$	abstraction
		$e e$	application

- Examples:

$$\begin{array}{cccc} & & x & y & z \\ & & \lambda x.x & \lambda x.y & \lambda x.\lambda y.x \\ x y & (\lambda x.x) z & x \lambda y.z & ((\lambda x.x) \lambda x.x) \end{array}$$

- Conventions when writing  $\lambda$ -expressions:
  - 1 Application associates to the left, e.g.,  $s t u = (s t) u$
  - 2 The body of an abstraction extends as far to the right as possible, e.g.,  $\lambda x.\lambda y.x y x = \lambda x.(\lambda y.((x y) x))$

## Bound and Free Variables

- An occurrence of variable  $x$  is said to be *bound* when it occurs inside  $\lambda x$ , otherwise said to be *free*.
  - ▶  $\lambda y.(x y)$
  - ▶  $\lambda x.x$
  - ▶  $\lambda z.\lambda x.\lambda x.(y z)$
  - ▶  $(\lambda x.x) x$
- Expressions without free variables is said to be *closed expressions* or *combinators*.

# Evaluation

To evaluate  $\lambda$ -expression  $e$ ,

- 1 Find a sub-expression of the form:

$$(\lambda x.e_1) e_2$$

Expressions of this form are called “redex” (reducible expression).

- 2 Rewrite the expression by substituting the  $e_2$  for every free occurrence of  $x$  in  $e_1$ :

$$(\lambda x.e_1) e_2 \rightarrow [x \mapsto e_2]e_1$$

This rewriting is called  $\beta$ -reduction

Repeat the above two steps until there are no redexes.

# Evaluation

- $\lambda x.x$
- $(\lambda x.x) y$
- $(\lambda x.x y)$
- $(\lambda x.x y) z$
- $(\lambda x.(\lambda y.x)) z$
- $(\lambda x.(\lambda x.x)) z$
- $(\lambda x.(\lambda y.x)) y$
- $(\lambda x.(\lambda y.x y)) (\lambda x.x) z$

# Substitution

The definition of  $[x \mapsto e_1]e_2$ :

$$\begin{aligned} [x \mapsto e_1]x &= e_1 \\ [x \mapsto e_1]y &= y \\ [x \mapsto e_1](\lambda y.e_2) &= \lambda z.[x \mapsto e_1]([y \mapsto z]e_2) \quad (\text{new } z) \\ [x \mapsto e_1](e_2 e_3) &= ([x \mapsto e_1]e_2 [x \mapsto e_1]e_3) \end{aligned}$$



# Evaluation Strategy

- In a lambda expression, multiple redexes may exist. Which redex to reduce next?

$$\lambda x.x (\lambda x.x (\lambda z.(\lambda x.x) z)) = id (id (\lambda z.id z))$$

redexes:

$$\underline{id (id (\lambda z.id z))}$$

$$id (\underline{id (\lambda z.id z)})$$

$$id (id (\lambda z.\underline{id z}))$$

- Evaluation strategies:
  - ▶ Normal order
  - ▶ Call-by-name
  - ▶ Call-by-value

## Normal order strategy

Reduce the leftmost, outermost redex first:

$$\begin{aligned} & id (id (\lambda z.id z)) \\ \rightarrow & \frac{id (id (\lambda z.id z))}{id (\lambda z.id z)} \\ \rightarrow & \lambda z.id z \\ \rightarrow & \lambda z.z \\ \not\rightarrow & \end{aligned}$$

The evaluation is deterministic (i.e., partial function).

## Call-by-name strategy

Follow the normal order reduction, not allowing reductions inside abstractions:

$$\begin{aligned} & id (id (\lambda z.id z)) \\ \rightarrow & \frac{id (\lambda z.id z)}{id (\lambda z.id z)} \\ \rightarrow & \lambda z.id z \\ \not\rightarrow & \end{aligned}$$

The call-by-name strategy is *non-strict* (or *lazy*) in that it evaluates arguments that are actually used.

## Call-by-value strategy

Reduce the outermost redex whose right-hand side has a *value* (a term that cannot be reduced any further):

$$\begin{aligned} & id (id (\lambda z.id z)) \\ \rightarrow & \frac{id (\lambda z.id z)}{} \\ \rightarrow & \lambda z.id z \\ \not\rightarrow & \end{aligned}$$

The call-by-name strategy is *strict* in that it always evaluates arguments, whether or not they are used in the body.

# Compiling to Lambda Calculus

Consider the source language:

$$\begin{array}{l} E \rightarrow \textit{true} \\ | \textit{false} \\ | n \\ | x \\ | E + E \\ | \textit{iszero } E \\ | \textit{if } E \textit{ then } E \textit{ else } E \\ | \textit{let } x = E \textit{ in } E \\ | \textit{letrec } f(x) = E \textit{ in } E \\ | \textit{proc } x E \\ | E E \end{array}$$

Define the translation procedure from  $E$  to  $\lambda$ -calculus.

# Compiling to Lambda Calculus

$E$ : the translation result of  $E$  in  $\lambda$ -calculus

$$\underline{true} = \lambda t. \lambda f. t$$

$$\underline{false} = \lambda t. \lambda f. f$$

$$\underline{0} = \lambda s. \lambda z. z$$

$$\underline{1} = \lambda s. \lambda z. (s z)$$

$$\underline{n} = \lambda s. \lambda z. (s^n z)$$

$$\underline{x} = x$$

$$\underline{E_1 + E_2} = (\lambda n. \lambda m. \lambda s. \lambda z. m s (n s z)) \underline{E_1} \underline{E_2}$$

$$\underline{iszero E} = (\lambda m. m (\lambda x. \underline{false}) \underline{true}) \underline{E}$$

$$\underline{\text{if } E_1 \text{ then } E_2 \text{ else } E_3} = \underline{E_1} \underline{E_2} \underline{E_3}$$

$$\underline{\text{let } x = E_1 \text{ in } E_2} = (\lambda x. \underline{E_2}) \underline{E_1}$$

$$\underline{\text{letrec } f(x) = E_1 \text{ in } E_2} = \underline{\text{let } f = Y (\lambda f. \lambda x. E_1) \text{ in } E_2}$$

$$\underline{\text{proc } x E} = \lambda x. \underline{E}$$

$$\underline{E_1 E_2} = \underline{E_1} \underline{E_2}$$

# Correctness of Compilation

## Theorem

For any expression  $E$ ,

$$\llbracket \underline{E} \rrbracket = \underline{\llbracket E \rrbracket}$$

where  $\llbracket E \rrbracket$  denotes the value that results from evaluating  $E$ .

## Examples: Booleans

$$\begin{aligned}\underline{\text{if } true \text{ then } 0 \text{ else } 1} &= \underline{true\ 0\ 1} \\ &= (\lambda t.\lambda f.t)\ \underline{0}\ \underline{1} \\ &= \underline{0} \\ &= \lambda s.\lambda z.z\end{aligned}$$

Note that

$$\llbracket \underline{\text{if } true \text{ then } 0 \text{ else } 1} \rrbracket = \llbracket \underline{\text{if } true \text{ then } 0 \text{ else } 1} \rrbracket$$



## Exercises

Define the translation for the boolean operations:

- $E_1$  and  $E_2$  =
- $E_1$  or  $E_2$  =
- not  $E$  =

## Example: Numerals

$$\begin{aligned}\underline{1 + 2} &= (\lambda n.\lambda m.\lambda s.\lambda z.m\ s\ (n\ s\ z))\ \underline{1}\ \underline{2} \\ &= \lambda s.\lambda z.\underline{2}\ s\ (\underline{1}\ s\ z) \\ &= \lambda s.\lambda z.\underline{2}\ s\ (\lambda s.\lambda z.(s\ z)\ s\ z) \\ &= \lambda s.\lambda z.\underline{2}\ s\ (s\ z) \\ &= \lambda s.\lambda z.(\lambda s.\lambda z.(s\ (s\ z)))\ s\ (s\ z) \\ &= \lambda s.\lambda z.s\ (s\ (s\ z)) \\ &= \underline{3}\end{aligned}$$

# Exercises

Define the translation for the boolean operations:

- $\text{succ } \underline{E} =$

- $\text{pred } \underline{E} =$

- $\underline{E_1 * E_2} =$

- $\underline{E_1^{E_2}} =$

# Recursion

- For example, the factorial function

$$f(n) = \text{if } n = 0 \text{ then } 1 \text{ else } n * f(n - 1)$$

is encoded by

$$\text{fact} = Y(\lambda f. \lambda n. \text{if } n = 0 \text{ then } 1 \text{ else } n * f(n - 1))$$

where  $Y$  is the Y-combinator (or fixed point combinator):

$$Y = \lambda f. (\lambda x. f (x x)) (\lambda x. f (x x))$$

- Then,  $\text{fact } n$  computes  $n!$ .
- Recursive functions can be encoded by composing non-recursive functions!

## Recursion

Let  $F = \lambda f.\lambda n.\text{if } n = 0 \text{ then } 1 \text{ else } n * f(n - 1)$  and  
 $G = \lambda x.F(x x)$ .

fact 1

$$= (Y F) 1$$

$$= (\lambda f.((\lambda x.f(x x))(\lambda x.f(x x)))) F) 1$$

$$= ((\lambda x.F(x x))(\lambda x.F(x x))) 1$$

$$= (G G) 1$$

$$= (F (G G)) 1$$

$$= (\lambda n.\text{if } n = 0 \text{ then } 1 \text{ else } n * (G G)(n - 1)) 1$$

$$= \text{if } 1 = 0 \text{ then } 1 \text{ else } 1 * (G G)(1 - 1)$$

$$= \text{if false then } 1 \text{ else } 1 * (G G)(1 - 1)$$

$$= 1 * (G G)(1 - 1)$$

$$= 1 * (F (G G))(1 - 1)$$

$$= 1 * (\lambda n.\text{if } n = 0 \text{ then } 1 \text{ else } n * (G G)(n - 1))(1 - 1)$$

$$= 1 * \text{if } (1 - 1) = 0 \text{ then } 1 \text{ else } (1 - 1) * (G G)((1 - 1) - 1)$$

$$= 1 * 1$$

# Summary

Programming language = Lambda calculus + Syntactic sugars

- $\lambda$ -calculus is a minimal programming language.
  - ▶ Syntax:  $e \rightarrow x \mid \lambda x.e \mid e e$
  - ▶ Semantics:  $\beta$ -reduction
- Yet,  $\lambda$ -calculus is Turing-complete.

$$\begin{array}{l} e \rightarrow x \\ \quad \mid \lambda x.e \\ \quad \mid e e \end{array} =$$

