COSE212: Programming Languages

Lecture 9 — Type System (1)

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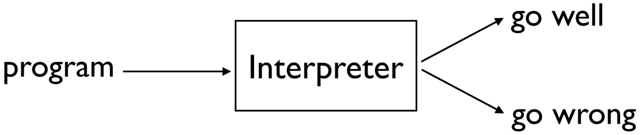
Program Execution without Type Checker



- `if true then 88 else 99`
- `if 3 then 88 else 99`
- `(proc (x) (x 3)) (proc (x) x)`
- `(proc (x) (x 3)) 4`
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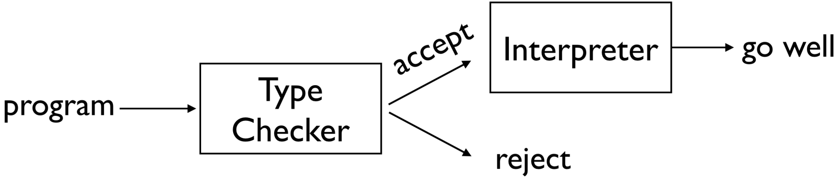

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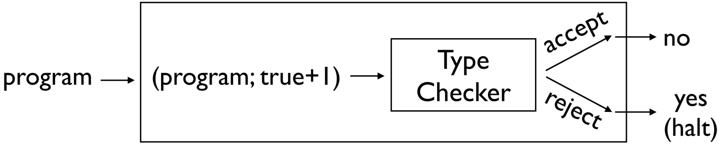
Type Checking is Undecidable

If possible, we can solve the halting problem using the type checker:



Only Approximate Type Checking is Possible

All static type systems in use give approximated answers, which can be divided into *sound* and *unsound* type checkers.

- Sound type checker guarantees that type checked programs do not go wrong. Used in ML, Haskell, etc.
- Unsound type checker does not give such guarantee. Used in C, C++, etc.

Plan: Sound Type Checker for the PROC Language

$$
\begin{array}{rcl}
E & \to & n \\
  & | & x \\
  & | & E + E \\
  & | & E - E \\
  & | & \texttt{iszero}\ E \\
  & | & \texttt{if}\ E\ \texttt{then}\ E\ \texttt{else}\ E \\
  & | & \texttt{let}\ x = E\ \texttt{in}\ E \\
  & | & \texttt{proc}\ x\ E \\
  & | & E\ E
\end{array}
$$

Plan: Sound Type Checker for the PROC Language

$$\frac{}{\rho \vdash n \Rightarrow n} \qquad \frac{}{\rho \vdash x \Rightarrow \rho(x)} \qquad \frac{\rho \vdash E_1 \Rightarrow n_1 \qquad \rho \vdash E_2 \Rightarrow n_2}{\rho \vdash E_1 + E_2 \Rightarrow n_1 + n_2}$$

$$\frac{\rho \vdash E \Rightarrow 0}{\rho \vdash \texttt{iszero}\ E \Rightarrow \mathit{true}} \qquad \frac{\rho \vdash E \Rightarrow n}{\rho \vdash \texttt{iszero}\ E \Rightarrow \mathit{false}}\ n \neq 0$$

$$\frac{\rho \vdash E_1 \Rightarrow \mathit{true} \qquad \rho \vdash E_2 \Rightarrow v}{\rho \vdash \texttt{if}\ E_1\ \texttt{then}\ E_2\ \texttt{else}\ E_3 \Rightarrow v} \qquad \frac{\rho \vdash E_1 \Rightarrow \mathit{false} \qquad \rho \vdash E_3 \Rightarrow v}{\rho \vdash \texttt{if}\ E_1\ \texttt{then}\ E_2\ \texttt{else}\ E_3 \Rightarrow v}$$

$$\frac{\rho \vdash E_1 \Rightarrow v_1 \qquad [x \mapsto v_1]\rho \vdash E_2 \Rightarrow v}{\rho \vdash \texttt{let}\ x = E_1\ \texttt{in}\ E_2 \Rightarrow v}$$

$$\frac{}{\rho \vdash \texttt{proc}\ x\ E \Rightarrow (x, E, \rho)}$$

$$\frac{\rho \vdash E_1 \vdash (x, E, \rho') \qquad \rho \vdash E_2 \Rightarrow v \qquad [x \mapsto v]\rho' \vdash E \Rightarrow v'}{\rho \vdash E_1\ E_2 \Rightarrow v'}$$

Types

Types are defined inductively:

$$
\begin{array}{rcl}
T & \to & \mathsf{int} \\
 & | & \mathsf{bool} \\
 & | & T \to T
\end{array}
$$

Examples:

- int
- bool
- int $\to$ int
- bool $\to$ int
- int $\to$ (int $\to$ bool)
- (int $\to$ int) $\to$ (bool $\to$ bool)
- (int $\to$ int) $\to$ (bool $\to$ (bool $\to$ int))

Types of Expressions

We need *type environment*:

$$\Gamma : Var \to T$$

Notation:

$$\Gamma \vdash e : t \Leftrightarrow \text{Under type environment } \Gamma\text{, expression } e \text{ has type } t.$$

Examples

- $[] \vdash \mathbf{3} : \mathsf{int}$
- $[x \mapsto \mathsf{int}] \vdash x : \mathsf{int}$
- $[] \vdash \mathbf{4} - \mathbf{3} :$
- $[x \mapsto \mathsf{int}] \vdash x - \mathbf{3} :$
- $[] \vdash \texttt{iszero } \mathbf{11} :$
- $[] \vdash \texttt{proc } (x)\ (x - \mathbf{11}) :$
- $[] \vdash \texttt{proc } (x)\ (\texttt{let } y = x - \mathbf{11} \texttt{ in } (x - y)) :$
- $[] \vdash \texttt{proc } (x)\ (\texttt{if } x \texttt{ then } \mathbf{11} \texttt{ else } \mathbf{22}) :$
- $[] \vdash \texttt{proc } (x)\ (\texttt{proc } (y) \texttt{ if } y \texttt{ then } x \texttt{ else } \mathbf{11}) :$
- $\texttt{proc } (f)\ (\texttt{if } (f\ \mathbf{3}) \texttt{ then } \mathbf{11} \texttt{ else } \mathbf{22}) :$
- $[] \vdash (\texttt{proc } (x)\ x)\ \mathbf{1} :$
- $[f \mapsto \mathsf{int} \to \mathsf{int}] \vdash (f\ (f\ \mathbf{1})) :$

Typing Rules

Inductive rules for assigning types to expressions:

$$\frac{}{\Gamma \vdash n : \mathsf{int}} \qquad \frac{}{\Gamma \vdash x : \Gamma(x)}$$

$$\frac{\Gamma \vdash E_1 : \mathsf{int} \qquad \Gamma \vdash E_2 : \mathsf{int}}{\Gamma \vdash E_1 + E_2 : \mathsf{int}} \qquad \frac{\Gamma \vdash E_1 : \mathsf{int} \qquad \Gamma \vdash E_2 : \mathsf{int}}{\Gamma \vdash E_1 - E_2 : \mathsf{int}}$$

$$\frac{\Gamma \vdash E : \mathsf{int}}{\Gamma \vdash \texttt{iszero}\ E : \mathsf{bool}} \qquad \frac{\Gamma \vdash E_1 : \mathsf{bool} \qquad \Gamma \vdash E_2 : t \qquad \Gamma \vdash E_3 : t}{\texttt{if}\ E_1\ \texttt{then}\ E_2\ \texttt{else}\ E_3 : t}$$

$$\frac{\Gamma \vdash E_1 : t_1 \qquad [x \mapsto t_1]\Gamma \vdash E_2 : t_2}{\Gamma \vdash \texttt{let}\ x = E_1\ \texttt{in}\ E_2 : t_2} \qquad \frac{\Gamma \vdash E_1 : t_1 \to t_2 \qquad \Gamma \vdash E_2 : t_1}{\Gamma \vdash E_1\ E_2 : t_2}$$

$$\frac{[x \mapsto t_1]\Gamma \vdash E : t_2}{\Gamma \vdash \texttt{proc}\ x\ E : t_1 \to t_2}$$

We say that a closed expression E has type t iff we can derive $[] \vdash E : t$.

Examples

- iszero $(\mathbf{1} + \mathbf{2})$
- proc (x) $(x - \mathbf{11})$
- proc (x) (if x then $\mathbf{11}$ else $\mathbf{22}$)
- (proc (x) x) $\mathbf{1}$
- proc (x) (proc (y) if y then x else $\mathbf{11}$)

Expressions May Have Multiple Types

- $\texttt{proc}\ x\ x$:

$$\frac{[x \mapsto \mathsf{int}] \vdash x : \mathsf{int}}{[] \vdash \texttt{proc}\ x\ x : \mathsf{int} \to \mathsf{int}}$$

$$\frac{[x \mapsto \mathsf{bool}] \vdash x : \mathsf{bool}}{[] \vdash \texttt{proc}\ x\ x : \mathsf{bool} \to \mathsf{bool}}$$

$$\frac{[x \mapsto (\mathsf{int} \to \mathsf{int})] \vdash x : \mathsf{int} \to \mathsf{int}}{[] \vdash \texttt{proc}\ x\ x : (\mathsf{int} \to \mathsf{int}) \to (\mathsf{int} \to \mathsf{int})}$$

- $\texttt{proc}\ (f)\ (f\ 3)$ has type $(\mathsf{int} \to t) \to t$ for any t.
- $\texttt{proc}\ (f)\ \texttt{proc}\ (x)\ (f\ (f\ x))$ has type $((t \to t) \to (t \to t))$ for any t.

Soundness of Typing Rules

Theorem (Soundness)

If a program E has a type error, we cannot find t such that $[] \vdash E : t$. In other words, If E is typed, i.e., $[] \vdash E : t$ for some t, then E is guaranteed to run well.

Examples:

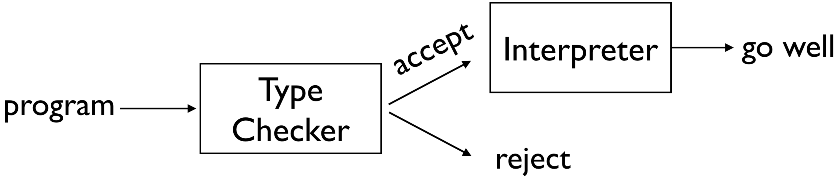
- `if 3 then 88 else 99`
- `(proc (x) (x 3)) 4`
- `let x = 4 in (x 3)`
- `(proc (x) (3 x))` e
- `let x = zero?  0 in (3-x)`
- $(\mathtt{proc}\ (x)\ (x+1))\ ((\mathtt{proc}\ y\ y)\ (\mathtt{proc}\ z\ z))$

Incompleteness

Even though some programs do not have type errors, they do not have types:

- $\texttt{if iszero } \mathbf{1} \texttt{ then } \mathbf{11} \texttt{ else } (\texttt{iszero } \mathbf{22}))$
- $(\texttt{proc } (f)\ (f\ f))\ (\texttt{proc } x\ x)$

Type Checker



- The type checker accepts a program E only if $[] \vdash E : t$ for some t.
- Otherwise, E is rejected.

Challenge

Given a program $\boldsymbol{E}$, how to check $[] \vdash \boldsymbol{E} : \boldsymbol{t}$? Nontrivial, because of the following type rule:

$$\frac{[x \mapsto t_1]\Gamma \vdash E : t_2}{\Gamma \vdash \texttt{proc}\ x\ E : t_1 \rightarrow t_2}$$

Two approaches:

- *Type Annotation*: Programmers are required to supply the type of the function argument. Used in C, C++, Java, etc.
- *Type Inference*: Type checker attempts to automatically infer types. Only possible if the language is carefully designed. Used in ML, Haskell, etc.