# COSE212: Programming Languages Lecture 1 — Inductive Definitions (1)

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#### Inductive Definitions

A technique for formally defining a set:

- The set is defined in terms of itself.
- The only way of defining an infinite set by a finite means.

Three styles to define a set inductively:

- Top-down
- Bottom-up
- Rules of inference

# Example (Top-Down)

#### Definition (S)

A natural number  $oldsymbol{n}$  is in  $oldsymbol{S}$  if and only if

- $oldsymbol{0}$  n=0, or
- $a n-3 \in S.$ 
  - Inductive definition of a set of natural numbers:  $S \subseteq \mathbb{N} = \{0, 1, \ldots\}$
  - $\{0,3,6,9,\ldots\}\subseteq S$
  - $\{0,3,6,9,\ldots\}\supseteq S$

$$S = \{0, 3, 6, 9, \ldots\}.$$

## Formal Proofs

#### Lemma

 $\{0,3,6,9,\ldots\}\subseteq S$ 

By induction. To show:  $3k \in S$  for all  $k \in \mathbb{N}$ .

$${f 0}$$
 Base case:  ${f 3}k\in S$  when  $k=0.$ 

2 Inductive case: Assume  $3k \in S$  (Induction Hypothesis, I.H.). To show is  $3 \cdot (k+1) \in S$ , which holds because  $3 \cdot (k+1) - 3 = 3k \in S$  by the induction hypothesis.

#### Lemma

 $\{0,3,6,9,\ldots\}\supseteq S$ 

By proof by contradiction. Let n = 3k + q (q = 1 or 2) and assume  $n \in S$ . By the definition of S, n - 3, n - 6,  $\ldots$ ,  $n - 3k \in S$ . Thus, S must include 1 or 2, a contradiction.

## A Bottom-up Definition

#### Definition (S)

S is the  $\mathit{smallest}$  set such that  $S \subseteq \mathbb{N}$  and S satisfies the following two conditions:

•  $0 \in S$ , and • if  $n \in S$ , then  $n + 3 \in S$ .

- The two conditions imply  $\{0,3,6,9,\ldots\}\subseteq S$ .
- The two conditions do not imply  $\{0,3,6,9,\ldots\}\supseteq S$ , e.g.,  $\mathbb N$ .
- By requiring S to be the smallest such a set,

$$S = \{0, 3, 6, 9, \ldots\}.$$

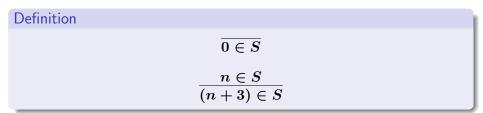
- The smallest set satisfying the conditions is unique.
  - ▶ Proof) If  $S_1$  and  $S_2$  satisfy the conditions and are both smallest, then  $S_1 \subseteq S_2$  and  $S_2 \subseteq S_1$ . Therefore,  $S_1 = S_2$  (⊆ is anti-symmetric).

#### **Rules of Inference**

# $rac{A}{B}$

- A: hypothesis (antecedent)
- B: conclusion (consequent)
- "if A is true then B is also true".
- $\overline{B}$ : axiom.

## Defining a Set by Rules of Inferences



Interpret the rules as follows:

"A natural number n is in S iff  $n \in S$  can be derived from the axiom by applying the inference rules finitely many times"

ex)  $3 \in S$  because

 $\overline{ \substack{0 \in S \\ 3 \in S}}$  the axiom the second rule

Note that this interpretation enforces that  $\boldsymbol{S}$  is the smallest set closed under the inference rules.

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### Summary

In inductive definitions, a set is defined in terms of itself. Three styles:

- Top-down
- Bottom-up
- Rules of inference

In PL, we mainly use the rules-of-inference method.

#### Exercises

• What set is defined by the following inductive rules?

$$\overline{3}$$
  $\frac{x \ y}{x+y}$ 

What set is defined by the following inductive rules?

$$\overline{()}$$
  $\frac{s}{(s)}$   $\frac{s}{ss}$ 

Of Define the following set as rules of inference:

 $S = \{a, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, bab, bba, bbb, \ldots\}$ 

Oefine the following set as rules of inference:

$$S = \{x^ny^{n+1} \mid n \in \mathbb{N}\}$$