COSE212: Programming Languages Lecture 1 — Inductive Definitions (1)

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Inductive Definitions

A technique for formally defining a set:

- The set is defined in terms of itself.
- The only way of defining an infinite set by a finite means.

Three styles to define a set inductively:

- Top-down
- Bottom-up
- Rules of inference

Example (Top-Down)

Definition (S)

A natural number $oldsymbol{n}$ is in $oldsymbol{S}$ if and only if

- $oldsymbol{0}$ n=0, or
- $a n-3 \in S.$
 - Inductive definition of a set of natural numbers: $S \subseteq \mathbb{N} = \{0, 1, \ldots\}$
 - $\{0,3,6,9,\ldots\}\subseteq S$
 - $\{0,3,6,9,\ldots\}\supseteq S$

$$S = \{0, 3, 6, 9, \ldots\}.$$

Formal Proofs

Lemma

 $\{0,3,6,9,\ldots\}\subseteq S$

By induction. To show: $3k \in S$ for all $k \in \mathbb{N}$.

$${f 0}$$
 Base case: ${f 3}k\in S$ when $k=0.$

2 Inductive case: Assume $3k \in S$ (Induction Hypothesis, I.H.). To show is $3 \cdot (k+1) \in S$, which holds because $3 \cdot (k+1) - 3 = 3k \in S$ by the induction hypothesis.

Lemma

 $\{0,3,6,9,\ldots\}\supseteq S$

By proof by contradiction. Let n = 3k + q (q = 1 or 2) and assume $n \in S$. By the definition of S, n - 3, n - 6, \ldots , $n - 3k \in S$. Thus, S must include 1 or 2, a contradiction.

A Bottom-up Definition

Definition (S)

S is the $\mathit{smallest}$ set such that $S \subseteq \mathbb{N}$ and S satisfies the following two conditions:

• $0 \in S$, and • if $n \in S$, then $n + 3 \in S$.

- The two conditions imply $\{0,3,6,9,\ldots\}\subseteq S$.
- The two conditions do not imply $\{0,3,6,9,\ldots\}\supseteq S$, e.g., $\mathbb N$.
- By requiring S to be the smallest such a set,

$$S = \{0, 3, 6, 9, \ldots\}.$$

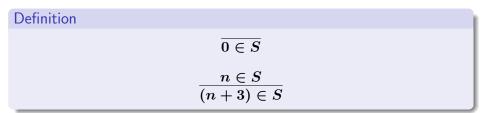
- The smallest set satisfying the conditions is unique.
 - ▶ Proof) If S_1 and S_2 satisfy the conditions and are both smallest, then $S_1 \subseteq S_2$ and $S_2 \subseteq S_1$. Therefore, $S_1 = S_2$ (⊆ is anti-symmetric).

Rules of Inference

$rac{A}{B}$

- A: hypothesis (antecedent)
- B: conclusion (consequent)
- "if A is true then B is also true".
- \overline{B} : axiom.

Defining a Set by Rules of Inferences



Interpret the rules as follows:

"A natural number n is in S iff $n \in S$ can be derived from the axiom by applying the inference rules finitely many times"

ex) $3 \in S$ because

 $\overline{ \substack{0 \in S \\ 3 \in S}}$ the axiom the second rule

Note that this interpretation enforces that \boldsymbol{S} is the smallest set closed under the inference rules.

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Summary

In inductive definitions, a set is defined in terms of itself. Three styles:

- Top-down
- Bottom-up
- Rules of inference

In PL, we mainly use the rules-of-inference method.

Exercises

• What set is defined by the following inductive rules?

$$\overline{3}$$
 $\frac{x \ y}{x+y}$

What set is defined by the following inductive rules?

$$\overline{()}$$
 $\frac{s}{(s)}$ $\frac{s}{ss}$

Of Define the following set as rules of inference:

 $S = \{a, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, bab, bba, bbb, \ldots\}$

Oefine the following set as rules of inference:

$$S = \{x^ny^{n+1} \mid n \in \mathbb{N}\}$$