COSE212: Programming Languages

Lecture 1 — Inductive Definitions (1)

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Inductive Definitions

A technique for formally defining a set:

- The set is defined in terms of itself.
- The only way of defining an infinite set by a finite means.

Three styles to define a set inductively:

- Top-down
- Bottom-up
- Rules of inference

Example (Top-Down)

Definition (S)

A natural number n is in S if and only if

1. $n = 0$, or
2. $n - 3 \in S$.

- Inductive definition of a set of natural numbers: $S \subseteq \mathbb{N} = \{0, 1, \ldots\}$
- $\{0, 3, 6, 9, \ldots\} \subseteq S$
- $\{0, 3, 6, 9, \ldots\} \supseteq S$

$$S = \{0, 3, 6, 9, \ldots\}.$$

Formal Proofs

Lemma

$\{0, 3, 6, 9, \ldots\} \subseteq S$

By induction. To show: $3k \in S$ for all $k \in \mathbb{N}$.

1. Base case: $3k \in S$ when $k = 0$.
2. Inductive case: Assume $3k \in S$ (Induction Hypothesis, I.H.). To show is $3 \cdot (k + 1) \in S$, which holds because $3 \cdot (k + 1) - 3 = 3k \in S$ by the induction hypothesis.

Lemma

$\{0, 3, 6, 9, \ldots\} \supseteq S$

By proof by contradiction. Let $n = 3k + q$ $(q = 1$ or $2)$ and assume $n \in S$. By the definition of S, $n - 3$, $n - 6$, $\ldots, n - 3k \in S$. Thus, S must include 1 or 2, a contradiction.

A Bottom-up Definition

Definition (S)

S is the *smallest* set such that $S \subseteq \mathbb{N}$ and S satisfies the following two conditions:

1. $0 \in S$, and
2. if $n \in S$, then $n + 3 \in S$.

- The two conditions imply $\{0, 3, 6, 9, \ldots\} \subseteq S$.
- The two conditions do not imply $\{0, 3, 6, 9, \ldots\} \supseteq S$, e.g., $\mathbb{N}$.
- By requiring S to be the **smallest** such a set,

$$S = \{0, 3, 6, 9, \ldots\}.$$

- The smallest set satisfying the conditions is unique.
  - Proof) If S_1 and S_2 satisfy the conditions and are both smallest, then $S_1 \subseteq S_2$ and $S_2 \subseteq S_1$. Therefore, $S_1 = S_2$ ($\subseteq$ is anti-symmetric).

Rules of Inference

$$\frac{A}{B}$$

- A: hypothesis (antecedent)
- B: conclusion (consequent)
- "if A is true then B is also true".
- $\overline{B}$: axiom.

Defining a Set by Rules of Inferences

Definition

$$\frac{}{\mathbf{0} \in \boldsymbol{S}}$$

$$\frac{\boldsymbol{n} \in \boldsymbol{S}}{(\boldsymbol{n} + \mathbf{3}) \in \boldsymbol{S}}$$

Interpret the rules as follows:

"A natural number $\boldsymbol{n}$ is in $\boldsymbol{S}$ iff $\boldsymbol{n} \in \boldsymbol{S}$ can be derived from the axiom by applying the inference rules finitely many times"

ex) $\mathbf{3} \in \boldsymbol{S}$ because

$$\cfrac{\overline{\mathbf{0} \in \boldsymbol{S}}\ \text{the axiom}}{\mathbf{3} \in \boldsymbol{S}}\ \text{the second rule}$$

Note that this interpretation enforces that $\boldsymbol{S}$ is the smallest set closed under the inference rules.

Summary

In inductive definitions, a set is defined in terms of itself. Three styles:

- Top-down
- Bottom-up
- Rules of inference

In PL, we mainly use the rules-of-inference method.

Exercises

1. What set is defined by the following inductive rules?

$$\frac{}{3} \qquad \frac{x \quad y}{x + y}$$

2. What set is defined by the following inductive rules?

$$\frac{}{()} \qquad \frac{s}{(s)} \qquad \frac{s}{s\ s}$$

3. Define the following set as rules of inference:

$$S = \{a, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, bab, bba, bbb, \ldots\}$$

4. Define the following set as rules of inference:

$$S = \{x^n y^{n+1} \mid n \in \mathbb{N}\}$$