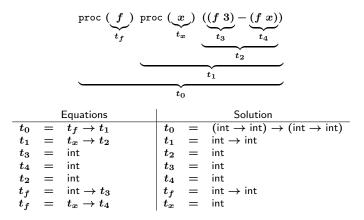
# COSE212: Programming Languages Lecture 11 — Automatic Type Inference (2)

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## Finding a Solution of Type Equations

Find values for the type variables that make all the equations true.



Such a solution can be found by the *unification algorithm*.

The calculation is split into equations to be solved and substitution found so far. Initially, the substitution is empty:

Equations	Substitution
$t_0 ~=~ t_f  ightarrow t_1$	
$t_1 \;=\; t_x  o t_2$	
$t_3$ = int	
$t_4$ = int	
$t_2$ = int	
$t_f \;=\; { m int}  o t_3$	
$t_f = t_x  o t_4$	

Consider each equation in turn. If the equation's left-hand side is a variable, move it to the substitution:

Equations	Substitution
$t_1 \;=\; t_x  o t_2$	$t_0 = t_f \rightarrow t_1$
$t_3$ = int	
$t_4$ = int	
$t_2$ = int	
$t_f \;=\; { m int}  o t_3$	
$t_f \;=\; t_x  o t_4$	

Move the next equation to the substitution and propagate the information to the existing substitution (i.e., substitute the right-hand side for each occurrence of  $t_1$ ):

		Equations		0	Substitution
$t_3$	=		$t_0$	=	$t_f \to (t_x \to t_2)$
$t_4$	=	int	$t_1$	=	$t_x  ightarrow t_2$
$t_2$	=	int			
$t_{f}$	=	$int \to t_3$			
$t_{f}$	=	$t_x  ightarrow t_4$			

Same for the next three equations:

Equations	Substitution
$t_4$ = int	$t_0 = t_f  ightarrow (t_x  ightarrow t_2)$
$t_2$ = int	$t_1 = t_x  ightarrow t_2$
$t_f =  ext{int}  o t_3$	$t_3 = int$
$t_f = t_x  o t_4$	
Equations	Substitution
$t_2 = \text{int}$	$t_0 = t_f  ightarrow (t_x  ightarrow t_2)$
$t_f \;\;=\;\; { m int}  o t_3$	$t_1 \hspace{.1in} = \hspace{.1in} t_x  ightarrow t_2$
$t_f = t_x  ightarrow t_4$	$t_3 = \text{int}$
-	$t_4$ = int
Equations	Substitution
$t_f = \operatorname{int}  ightarrow t_3$	$t_0 = t_f  ightarrow (t_x  ightarrow  ext{int})$
$t_f = t_x  ightarrow t_4$	$t_1 = t_x  o  ext{int}$
	$t_3 = int$
	$egin{array}{rcl} t_4&=& ext{int}\ t_2&=& ext{int} \end{array}$
	$t_2 = int$

Consider the next equation  $t_f = \text{int} \rightarrow t_3$ . The equation contains  $t_3$ , which is already bound to int in the substitution. Substitute int for  $t_3$  in the equation. This is called *applying* the substitution to the equation.

Equations	Substitution
$t_f = \text{int}  ightarrow  ext{int}$	$t_0 = t_f  ightarrow (t_x  ightarrow  ext{int})$
$t_f \;\;=\;\; t_x  o t_4$	$t_1 \;\;=\;\; t_x  o { ext{int}}$
	$t_3$ = int
	$t_4$ = int
	$\begin{array}{rcl} t_0 &=& t_f \rightarrow (t_x \rightarrow \operatorname{int}) \\ t_1 &=& t_x \rightarrow \operatorname{int} \\ t_3 &=& \operatorname{int} \\ t_4 &=& \operatorname{int} \\ t_2 &=& \operatorname{int} \end{array}$

Move the resulting equation to the substitution and update it.

Equations	Substitution	
$t_f = t_x  o t_4$	$t_0 = (\text{int} \rightarrow \text{int}) \rightarrow (t_x \rightarrow \text{int})$	-
	$t_1 = t_x  ightarrow  ext{int}$	
	$t_3 = \text{int}$	
	$t_4 = \text{int}$	
	$t_2 = \text{int}$	
	$\begin{array}{rcl}t_{0} & (\operatorname{int} f \operatorname{int} f) & f(t_{x} + \operatorname{int} f) \\ t_{1} & = & t_{x} \to \operatorname{int} \\ t_{3} & = & \operatorname{int} \\ t_{4} & = & \operatorname{int} \\ t_{2} & = & \operatorname{int} \\ t_{f} & = & \operatorname{int} \to \operatorname{int} \end{array}$	

Apply the substitution to the equation:

Equations	Substitution
$int  o int = t_x  o t_4$	$t_0 = (\operatorname{int}  ightarrow \operatorname{int})  ightarrow (t_x  ightarrow \operatorname{int})$
	$t_1 = t_x  ightarrow  ext{int}$
	$t_3 = \text{int}$
	$t_4$ = int
	$t_2 = int$
	$\begin{array}{rcl}t_0 &=& (\operatorname{int} \to \operatorname{int}) \to (t_x \to \operatorname{int})\\t_1 &=& t_x \to \operatorname{int}\\t_3 &=& \operatorname{int}\\t_4 &=& \operatorname{int}\\t_2 &=& \operatorname{int}\\t_f &=& \operatorname{int} \to \operatorname{int}\end{array}$

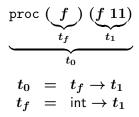
If neither side of the equation is a variable, simplify the equation by yielding two new equations:

Equations			Substitution
int = $t_x$	$t_0$	=	$(\text{int} \rightarrow \text{int}) \rightarrow (t_x \rightarrow \text{int})$
int = int	$t_1$	=	$t_x  ightarrow$ int
	$t_3$	=	int
	$t_4$	=	int
	$t_2$	=	int
	$t_{f}$	=	$\begin{array}{l} (\operatorname{int} \to \operatorname{int}) \to (t_x \to \operatorname{int}) \\ t_x \to \operatorname{int} \\ \operatorname{int} \\ \operatorname{int} \\ \operatorname{int} \\ \operatorname{int} \to \operatorname{int} \end{array}$

Switch the sides of the first equation and move it to the substitution:

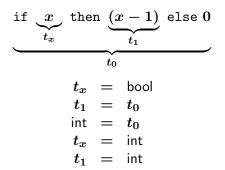
Equations		Substitution			
int	=	int	$t_0$	=	$(\text{int} \rightarrow \text{int}) \rightarrow (t_x \rightarrow \text{int})$ $\text{int} \rightarrow \text{int}$ int int int $\text{int} \rightarrow \text{int}$ int int
			$t_1$	=	int $\rightarrow$ int
			$t_3$	=	int
			$t_4$	=	int
			$t_2$	=	int
			$t_{f}$	=	int $\rightarrow$ int
			$t_x$	=	int

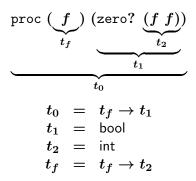
The final substitution is the solution of the original equations.



1 Substitution Equations  $t_0 = t_f \rightarrow t_1$  $t_f = \text{int} \rightarrow t_1$ 2 Equations Substitution  $t_f = \text{int} \rightarrow t_1$  $t_0 = t_f \rightarrow t_1$ 3 Equations Substitution  $egin{array}{rcl} t_0 &=& (\operatorname{int} 
ightarrow t_1) 
ightarrow t_1 \ t_f &=& \operatorname{int} 
ightarrow t_1 \end{array}$ 

The type is *polymorphic* in  $t_1$ .





#### Exercises

For each following expression, perform the type inference and find its type, or determine that no such type exists.

$$\textcircled{0} \texttt{ let } x = 4 \texttt{ in } (x \texttt{ 3})$$

2 let 
$$f = \text{proc}(z) \ z \text{ in proc}(x) \ ((f \ x) - 1))$$

() let p = zero? 1 in if p then 88 else 99

• let  $f = \operatorname{proc}(x) x$  in if  $(f (\operatorname{zero}?0))$  then  $(f \ 11)$  else  $(f \ 22)$ 

## Summary

Automatic type inference:

- derive type equations from the program text, and
- solve the equations by unification.