COSE212: Programming Languages

Lecture 1 — Inductive Definitions (1)

Hakjoo Oh
2015 Fall

Inductive Definitions

- A technique for formally defining a set.
- The set is defined in terms of itself.
- The only way of defining an infinite set by a finite means.

Example

Definition

A natural number n is in S if and only if

1. $n = 0$, or
2. $n - 3 \in S$.

What is the set S?

Example

Definition

A natural number n is in S if and only if

1. $n = 0$, or
2. $n - 3 \in S$.

What is the set S?

- $\{0, 3, 6, 9, \ldots\} \subseteq S$

Example

Definition

A natural number n is in S if and only if

1. $n = 0$, or
2. $n - 3 \in S$.

What is the set S?

- $\{0, 3, 6, 9, \ldots\} \subseteq S$
- $\{0, 3, 6, 9, \ldots\} \supseteq S$

A Bottom-up Version

Definition

S is the *smallest* set such that $S \subseteq \mathbb{N}$ and S satisfies the following two conditions:

1. $0 \in S$, and
2. if $n \in S$, then $n + 3 \in S$.

What is the set S?

A Bottom-up Version

Definition

S is the *smallest* set such that $S \subseteq \mathbb{N}$ and S satisfies the following two conditions:

1. $0 \in S$, and
2. if $n \in S$, then $n + 3 \in S$.

What is the set S?

- If the two conditions are satisfied, $\{0, 3, 6, 9, \ldots\} \subseteq S$.

A Bottom-up Version

Definition

S is the *smallest* set such that $S \subseteq \mathbb{N}$ and S satisfies the following two conditions:

1. $0 \in S$, and
2. if $n \in S$, then $n + 3 \in S$.

What is the set S?

- If the two conditions are satisfied, $\{0, 3, 6, 9, \ldots\} \subseteq S$.
- S is the **smallest** such a set.
- The smallest set is unique.

Rules of Inference

$$\frac{A}{B}$$

- A: hypothesis (antecedent)
- B: conclusion (consequent)
- "if A is true then B is also true".
- $\overline{B}$: axiom.

Defining a Set by Rules of Inferences

Definition

$$\overline{0 \in S}$$

$$\frac{n \in S}{(n+3) \in S}$$

Interpret the rules as follows:

"A natural number n is in S iff $n \in S$ can be derived from the axiom by applying the inference rules finitely many times"

Defining a Set by Rules of Inferences

Definition

$$\frac{}{\mathbf{0} \in \boldsymbol{S}}$$

$$\frac{\boldsymbol{n} \in \boldsymbol{S}}{(\boldsymbol{n} + \mathbf{3}) \in \boldsymbol{S}}$$

Interpret the rules as follows:

"A natural number $\boldsymbol{n}$ is in $\boldsymbol{S}$ iff $\boldsymbol{n} \in \boldsymbol{S}$ can be derived from the axiom by applying the inference rules finitely many times"

ex) $\mathbf{3} \in \boldsymbol{S}$ because

$$\cfrac{\cfrac{}{\mathbf{0} \in \boldsymbol{S}}\ \text{the axiom}}{\mathbf{3} \in \boldsymbol{S}}\ \text{the second rule}$$

Defining a Set by Rules of Inferences

Definition

$$\overline{0 \in S}$$

$$\frac{n \in S}{(n+3) \in S}$$

Interpret the rules as follows:

"A natural number n is in S iff $n \in S$ can be derived from the axiom by applying the inference rules finitely many times"

ex) $3 \in S$ because

$$\cfrac{\overline{0 \in S}\ \text{the axiom}}{3 \in S}\ \text{the second rule}$$

Note that this interpretation enforces that S is the smallest set closed under the inference rules.

Summary

In inductive definitions, a set is defined in terms of itself. Three styles:

- Top-down
- Bottom-up
- Rules of inference

In PL, we mainly use the rules-of-inference method.

Exercises

1. What set is defined by the following inductive rules?

$$\frac{}{\mathbf{3}} \qquad \frac{\boldsymbol{x} \quad \boldsymbol{y}}{\boldsymbol{x} + \boldsymbol{y}}$$

Exercises

1. What set is defined by the following inductive rules?

$$\overline{\mathbf{3}} \qquad \frac{x \quad y}{x + y}$$

2. What set is defined by the following inductive rules?

$$\overline{(\,)} \qquad \frac{s}{s\,(\,)} \qquad \frac{s}{(\,)\,s} \qquad \frac{s}{(s)}$$

Exercises

1. What set is defined by the following inductive rules?

$$\frac{}{3} \qquad \frac{x \quad y}{x+y}$$

2. What set is defined by the following inductive rules?

$$\frac{}{()} \qquad \frac{s}{s\,()} \qquad \frac{s}{()\,s} \qquad \frac{s}{(s)}$$

3. Define the following set as rules of inference:

$$S = \{a, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, bab, bba, bbb, \ldots\}$$

Exercises

1. What set is defined by the following inductive rules?

$$\frac{}{3} \qquad \frac{x \quad y}{x + y}$$

2. What set is defined by the following inductive rules?

$$\frac{}{()} \qquad \frac{s}{s\,()} \qquad \frac{s}{()\,s} \qquad \frac{s}{(s)}$$

3. Define the following set as rules of inference:

$$S = \{a, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, bab, bba, bbb, \ldots\}$$

4. Define the following set as rules of inference:

$$S = \{x^n y^{n+1} \mid n \in \mathbb{N}\}$$