AAA616: Program Analysis

Lecture 5 — Abstract Interpretation Framework

Hakjoo Oh 2022 Fall

Abstract Interpretation Framework

A powerful framework for designing correct static analysis

- "framework": correct static analysis comes out, reusable
- "powerful": all static analyses are understood in this framework
- "simple": prescription is simple
- "eye-opening": any static analysis is an abstract interpretation

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1. INTRODUCTION and Science

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Step 1: Define Concrete Semantics

The concrete semantics describes the real executions of the program. Described by semantic domain and function.

- A semantic domain **D**, which is a CPO:
 - D is a partially ordered set with a least element \perp .
 - Any increasing chain $d_0 \sqsubseteq d_1 \sqsubseteq \dots$ in D has a least upper bound $\bigsqcup_{n \ge 0} d_n$ in D.
- A semantic function F:D o D, which is continuous: for all chains $d_0\sqsubseteq d_1\sqsubseteq\ldots$,

$$F(\bigsqcup_{n\geq 0}d_i)=\bigsqcup_{n\geq 0}F(d_n).$$

Then, the concrete semantics (or collecting semantics) is defined as the least fixed point of semantic function $F: D \rightarrow D$:

$$fixF = \bigsqcup_{i \in N} F^i(\bot).$$

Step 2: Define Abstract Semantics

Define the abstract semantics of the input program.

- Define an abstract semantic domain CPO \hat{D} .
 - Intuition: \hat{D} is an abstraction of D
- Define an abstract semantic function $\hat{F}:\hat{D}
 ightarrow\hat{D}.$
 - Intuition: \hat{F} is an abstraction of F.
 - \hat{F} must be monotone:

$$orall \hat{x}, \hat{y} \in \hat{D}. \ \hat{x} \sqsubseteq \hat{y} \implies \hat{F}(\hat{x}) \sqsubseteq \hat{F}(\hat{y})$$

(or extensive:
$$orall x \in \hat{D}. \ x \sqsubseteq \hat{F}(x))$$

Then, static analysis is to compute an upper bound of:

$$igsqcup_{i\in\mathbb{N}}\hat{F}^i(ot)$$

How can we ensure that the result soundly approximate the concrete semantics?

Requirement 1: Galois Connection

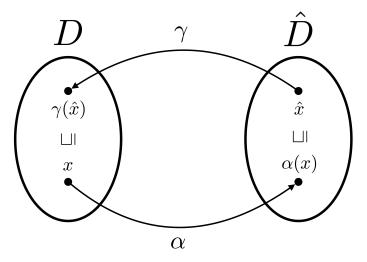
D and \hat{D} must be related with Galois-connection:

$$D \stackrel{\gamma}{\underset{\alpha}{\longleftarrow}} \hat{D}$$

That is, we have

- abstraction function: $lpha\in D o \hat{D}$
 - \blacktriangleright represents elements in D as elements of \hat{D}
- concretization function: $\gamma\in\hat{D} o D$
 - lacksim gives the meaning of elements of \hat{D} in terms of D
- $\forall x \in D, \hat{x} \in \hat{D}. \ \alpha(x) \sqsubseteq \hat{x} \iff x \sqsubseteq \gamma(\hat{x})$
 - lpha and γ respect the orderings of D and \hat{D}
 - If an element x ∈ D is safely described by x̂ ∈ D̂, i.e., α(d) ⊑ d̂, then the element described by x̂ is also safe w.r.t. x, i.e., x ⊑ γ(x̂)

Galois-Connection



Example: Sign Abstraction

$$\wp(\mathbb{Z}) \stackrel{\gamma}{\underbrace{\longleftrightarrow}} (\{\bot, +, 0, -, \top\}, \sqsubseteq)$$
$$\alpha(Z) = \begin{cases} \bot & Z = \emptyset \\ + & \forall z \in Z. \ z > 0 \\ 0 & Z = \{0\} \\ - & \forall z \in Z. \ z < 0 \\ \top & \text{otherwise} \end{cases}$$
$$\gamma(\bot) = \emptyset$$
$$\gamma(\top) = \mathbb{Z}$$
$$\gamma(+) = \{z \in \mathbb{Z} \mid z > 0\}$$
$$\gamma(0) = \{0\}$$
$$\gamma(-) = \{z \in \mathbb{Z} \mid z < 0\}$$

Example: Interval Abstraction

$$egin{aligned} \wp(\mathbb{Z}) & \stackrel{\gamma}{\underset{\alpha}{\longleftarrow}} \{\bot\} \cup \{[a,b] \mid a \in \mathbb{Z} \cup \{-\infty\}, b \in \mathbb{Z} \cup \{+\infty\}\} \ & \gamma(\bot) &= \emptyset \ & \gamma([a,b]) &= \{z \in \mathbb{Z} \mid a \leq z \leq b\} \ & lpha(\emptyset) &= \bot \ & lpha(X) &= [\min X, \max X] \end{aligned}$$

cf) Alternate Formulation

D and \hat{D} are related with Galois-connection:

$$D \stackrel{\gamma}{\underset{\alpha}{\longleftarrow}} \hat{D}$$

iff (α, γ) satisfies the following conditions:

- lpha and γ are monotone functions
- $\gamma \circ \alpha$ is extensive, i.e., $\gamma \circ \alpha \sqsupseteq \lambda x.x$
 - abstraction typically loses precision
 - $(\gamma \circ \alpha)(\{1,3\}) = \{1,2,3\}$
- $\alpha \circ \gamma$ is reductive: i.e., $\alpha \circ \gamma \sqsubseteq \lambda x.x$
 - If $\alpha \circ \gamma = \lambda x . x$, Galois-insertion.
 - With Galois-insertion, no two abstract elements describe the same concrete element, which may be true with Galois-connection.

$\mathsf{Proof}\ (\Rightarrow)$

If we have a Galois-connection:

$$orall x \in D, \hat{x} \in \hat{D}. \ lpha(x) \sqsubseteq \hat{x} \iff x \sqsubseteq \gamma(\hat{x})$$

then

- $\lambda x.x \sqsubseteq \gamma \circ \alpha$: $\alpha(x) \sqsubseteq \alpha(x)$ and hence $x \sqsubseteq \gamma(\alpha(x))$ by Galois-connection.
- $\alpha \circ \gamma \sqsubseteq \lambda x.x$: $\gamma(\hat{x}) \sqsubseteq \gamma(\hat{x})$ and hence $\alpha(\gamma(\hat{x})) \sqsubseteq \hat{x}$ by Galois-connection.
- γ is monotone: if $\hat{x} \sqsubseteq \hat{y}$, then $\alpha(\gamma(\hat{x})) \sqsubseteq \hat{y}$. Hence $\gamma(\hat{x}) \sqsubseteq \gamma(\hat{y})$ by Galois-connection.
- α is monotone: if $x \sqsubseteq y$, then $x \sqsubseteq \gamma(\alpha(y))$. Hence $\alpha(x) \sqsubseteq \alpha(y)$ by Galois-connection.

$\mathsf{Proof}(\Leftarrow)$

- Assume $\alpha(x) \sqsubseteq \hat{x}$. Since γ is monotone, $\gamma(\alpha(x)) \sqsubseteq \gamma(\hat{x})$. Because $\gamma \circ \alpha$ is extensive, we have $x \sqsubseteq \gamma(\hat{x})$.
- Assume $x \sqsubseteq \gamma(\hat{x})$. Since α is monotone, $\alpha(x) \sqsubseteq \alpha(\gamma(\hat{x}))$. Because $\alpha \circ \gamma$ is reductive, we have $\alpha(x) \sqsubseteq \hat{x}$.

Properties of Galois-Connection (1)

Given
$$D \stackrel{\gamma}{\longleftrightarrow lpha} \hat{D}$$
, we have:

•
$$\gamma \circ \alpha \circ \gamma = \gamma$$

- From α ∘ γ ⊑ λx.x and monotonicity of γ, we have γ ∘ α ∘ γ ⊑ γ.
 We have γ ∘ α ∘ γ ⊒ γ from γ ∘ α ⊒ λx.x.
- $\alpha \circ \gamma \circ \alpha = \alpha$
- $\alpha \circ \gamma$ and $\gamma \circ \alpha$ are idempotent:

$$(\alpha \circ \gamma)^2 = \alpha \circ \gamma, (\gamma \circ \alpha)^2 = \gamma \circ \alpha$$

• γ uniquely determines $\alpha(D, \hat{D} \text{ complete lattices})$:

$$\alpha(d) = \bigcap \{ \hat{d} \mid d \sqsubseteq \gamma(\hat{d}) \}$$

which implies that $\alpha(d)$ is the best abstraction of d.

• α uniquely determines γ :

$$\gamma(\hat{d}) = \bigsqcup \{ d \mid \alpha(d) \sqsubseteq \hat{d} \}$$

Properties of Galois-Connection (2)

- α is strict, i.e., $\alpha(\perp) = \hat{\perp}$. Proof. From $\perp \sqsubseteq \gamma(\hat{\perp})$, we have $\alpha(\perp) \sqsubseteq \hat{\perp}$ by Galois-connection.
- lpha is continuous: for any chain S in D,

$$lpha(\bigsqcup_{x\in S}x)=\bigsqcup_{x\in S}lpha(x).$$

Proof. Since α is monotonic,

$$\bigsqcup_{x\in S}lpha(x)\sqsubseteqlpha(\bigsqcup_{x\in S}x).$$

Since $\lambda x.x \sqsubseteq \gamma \circ \alpha$ and γ is monotonic,

$$\bigsqcup_{x\in S} x \sqsubseteq \bigsqcup_{x\in S} \gamma(\alpha(x)) \sqsubseteq \gamma(\bigsqcup_{x\in S} \alpha(x))$$

By Galois-connection, we have

$$lpha(\bigsqcup_{x\in S}x)\sqsubseteq\bigsqcup_{x\in S}lpha(x)$$

Deriving Galois-Connections

• Pointwise lifting: Given $D \xleftarrow{\gamma}{lpha} \hat{D}$ and a set S, then

$$S o D \xleftarrow{\gamma'}{\alpha'} S o \hat{D}$$

with $\alpha'(f) = \lambda s \in S.\alpha(f(s))$ and $\gamma(f) = \lambda s \in S.\gamma(f(s))$. • Composition: Given $X_1 \xrightarrow{\gamma_1} X_2 \xrightarrow{\gamma_2} X_3$, we have

$$X_1 \xleftarrow{\gamma_1 \circ \gamma_2}{\alpha_2 \circ \alpha_1} X_3$$

Requirement 2: \hat{F} and F

•
$$\hat{F}$$
 is a sound abstraction of F :

$$F\circ\gamma\sqsubseteq\gamma\circ\hat{F}\quad(lpha\circ F\sqsubseteq\hat{F}\circlpha)$$

• or, alternatively,

$$lpha(x) \sqsubseteq \hat{x} \implies lpha(F(x)) \sqsubseteq \hat{F}(\hat{x})$$

Best Abstract Semantics
From
$$D \xrightarrow{\gamma} \hat{D}$$
 and $F \circ \gamma \sqsubseteq \gamma \circ \hat{F}$, we have
 $\alpha \circ F \circ \gamma \sqsubseteq \alpha \circ \gamma \circ \hat{F}$ α is monotone
 $\sqsubseteq \hat{F}$ $\alpha \circ \gamma \sqsubseteq \lambda x.x$

The result means that $\alpha \circ F \circ \gamma$ is the best abstraction of F and any sound abstraction \hat{F} of F is greater than $\alpha \circ F \circ \gamma$.

Composition

When F, F' are concrete operators and \hat{F}, \hat{F}' are abstract operators, if \hat{F} and \hat{F}' are sound abstractions of F and F', respectively, then $\hat{F} \circ \hat{F}'$ is a sound abstraction of $F \circ F'$.

Fixpoint Transfer Theorems

Theorem (Fixpoint Transfer)

Let D and \hat{D} be related by Galois-connection $D \xleftarrow{\gamma}{\alpha} \hat{D}$. Let $F : D \to D$ be a continuous function and $\hat{F} : \hat{D} \to \hat{D}$ be a monotone function such that $\alpha \circ F \sqsubseteq \hat{F} \circ \alpha$. Then,

$$lpha(\mathit{fix}F) \sqsubseteq \bigsqcup_{i \in \mathbb{N}} \hat{F}^i(\hat{\perp}).$$

Theorem (Fixpoint Transfer2)

Let D and \hat{D} be related by Galois-connection $D \xleftarrow{\gamma}{\alpha} \hat{D}$. Let $F: D \to D$ be a continuous function and $\hat{F}: \hat{D} \to \hat{D}$ be a monotone function such that $\alpha(x) \sqsubseteq \hat{x} \implies \alpha(F(x)) \sqsubseteq \hat{F}(\hat{x})$. Then,

$$lpha(\mathit{fix}F) \sqsubseteq \bigsqcup_{i \in \mathbb{N}} \hat{F}^i(\hat{\perp}).$$

Proof of Fixpoint Transfer

• From $\alpha \circ F \sqsubseteq \hat{F} \circ \alpha$, we can derive

 $\forall n \in \mathbb{N}. \ \alpha \circ F^n \sqsubseteq \hat{F}^n \circ \alpha \quad (\forall n \in \mathbb{N}. \ \alpha(F^n(\bot)) \sqsubseteq \hat{F}^n(\hat{\bot}))$

by induction as follows:

$$\begin{array}{rcl} \alpha \circ F^{n+1} & = & \alpha \circ F \circ F^n \\ & \sqsubseteq & \alpha \circ F \circ \gamma \circ \alpha \circ F^n & \cdots \alpha \circ F \text{ is mono. and } \lambda x.x \sqsubseteq \gamma \circ \alpha \\ & \sqsubseteq & \alpha \circ F \circ \gamma \circ \hat{F}^n \circ \alpha & \cdots \alpha \circ F \circ \gamma \text{ is mono. and by I.H.} \\ & \sqsubseteq & \hat{F} \circ \hat{F}^n \circ \alpha & \cdots \alpha \circ F \circ \gamma \sqsubseteq \hat{F} \end{array}$$

• Since $lpha,F,\hat{F}$ are monotone, $\{lpha(F^i(otn))\}_i$ and $\{\hat{F}^i(otn)\}_i$ are chains, and

$$\bigsqcup_{i\in\mathbb{N}}\alpha(F^{i}(\bot))\sqsubseteq\bigsqcup_{i\in\mathbb{N}}\hat{F}^{i}(\hat{\bot})$$
(1)

• Since α and F are continuous,

$$\bigsqcup_{i\in\mathbb{N}}lpha(F^i(\bot))=lpha(\bigsqcup_{i\in\mathbb{N}}(F^i(\bot)))=lpha(\mathit{fix}F)$$

By replacing the left-hand side of (1), we have

$$lpha(\mathit{fix}F) \sqsubseteq igsqcup_{i\in\mathbb{N}}\hat{F}^i(\hat{ot})$$

Computing $igsqcup_{i\in\mathbb{N}}\hat{F}^{i}(\hat{ot})$

• If the abstract domain \hat{D} has finite height (i.e., all chains are finite), we can directly calculate

$$igsqcup_{i\in\mathbb{N}}\hat{F}^{i}(\hat{ot}).$$

$$igsqcup_{i\in\mathbb{N}}\hat{F}^i(\hat{ot})\sqsubseteq \lim_{i\in\mathbb{N}}\hat{X}_i$$

Fixpoint Accerlation with Widening

Define finite chain \hat{X}_i by an widening operator $\nabla: \hat{D} \times \hat{D} \rightarrow \hat{D}$:

$$\begin{aligned} \hat{X}_0 &= \hat{\perp} \\ \hat{X}_i &= \hat{X}_{i-1} & \text{if } \hat{F}(\hat{X}_{i-1}) \sqsubseteq \hat{X}_{i-1} \\ &= \hat{X}_{i-1} \bigtriangledown \hat{F}(\hat{X}_{i-1}) & \text{otherwise} \end{aligned}$$
 (2)

Conditions on ∇ :

• $\forall a,b\in \hat{D}.\;(a\sqsubseteq a\bigtriangledown b)\;\wedge\;(b\sqsubseteq a\bigtriangledown b)$

• For all increasing chains $(x_i)_i$, the increasing chain $(y_i)_i$ defined as

$$y_i = \left\{egin{array}{cc} x_0 & ext{if } i=0 \ y_{i-1}ig x_i & ext{if } i>0 \end{array}
ight.$$

eventually stabilizes (i.e., the chain is finite).

Decreasing Iterations with Narrowing

- We can refine the widening result $\lim_{i\in\mathbb{N}}\hat{X}_i$ by a narrowing operator $\hat{\Delta}:\hat{D}\times\hat{D}\to\hat{D}$.
- Compute chain $(\hat{Y}_i)_i$

$$\hat{Y}_{i} = \begin{cases} \lim_{i \in \mathbb{N}} \hat{X}_{i} & \text{if } i = 0\\ \hat{Y}_{i-1} \bigtriangleup \hat{F}(\hat{Y}_{i-1}) & \text{if } i > 0 \end{cases}$$
(3)

- Conditions on ∆
 - $\blacktriangleright \, \forall a,b \in \hat{D}. \ a \sqsubseteq b \implies a \sqsubseteq a \bigtriangleup b \sqsubseteq b$
 - \blacktriangleright For all decreasing chain $(x_i)_i$, the decreasing chain $(y_i)_i$ defined as

$$y_i = \left\{egin{array}{cc} x_i & ext{if } \mathrm{i} = 0 \ y_{i-1} igtriangleq x_i & ext{if } i > 0 \end{array}
ight.$$

eventually stabilizes.

Safety of Widening and Narrowing

Theorem (Widening's Safety)

Let \hat{D} be a CPO, $\hat{F} : \hat{D} \to \hat{D}$ a monotone function, $\nabla : \hat{D} \times \hat{D} \to \hat{D}$ a widening operator. Then, chain $(\hat{X}_i)_i$ defined as (2) eventually stabilizes and

$$igsqcup_{\in\mathbb{N}}\hat{F}^i(\hat{ot})\sqsubseteq \lim_{i\in\mathbb{N}}\hat{X}_i.$$

Theorem (Narrowing's Safety)

Let \hat{D} be a CPO, $\hat{F} : \hat{D} \to \hat{D}$ a monotone function, $\Delta : \hat{D} \times \hat{D} \to \hat{D}$ a narrowing operator. Then, chain $(\hat{Y}_i)_i$ defined as (3) eventually stabilizes and

$$igsqcup_{i\in\mathbb{N}}\hat{F}^i(\hat{ot})\sqsubseteq \lim_{i\in\mathbb{N}}\hat{Y}_i.$$

Proof of Widening's Safety

• We first show that $\{\hat{F}(\hat{X}_i)\}_i$ is an increasing chain (if so, by the second condition of widening, the widening sequence $\{\hat{X}_i\}_i$ eventually stabilizes). Note that, by (2), $\hat{F}(\hat{X}_{i+1})$ is either $\hat{F}(\hat{X}_i)$ or $\hat{F}(\hat{X}_i \bigtriangledown \hat{F}(\hat{X}_i))$. Since $\hat{X}_i \sqsubseteq \hat{X}_i \bigtriangledown \hat{F}(\hat{X}_i)$ and \hat{F} is monotone, for all i we have

$$\hat{F}(\hat{X}_i) \sqsubseteq \hat{F}(\hat{X}_{i+1})$$

- We next show that $orall i \in \mathbb{N}.\ \hat{F}^i(\hat{\perp}) \sqsubseteq \hat{X}_i.$
 - Base case. $\hat{F}^0(\perp) = \hat{\perp} \sqsubseteq \hat{X}_0.$
 - ▶ Inductive case. From the induction hypothesis (I.H.), i.e., $\hat{F}^i(\hat{\perp}) \sqsubseteq \hat{X}_i$, and the monotonicity of \hat{F} , we have

$$\hat{F}^{i+1}(\hat{\perp}) \sqsubseteq \hat{F}(\hat{X}_i) \tag{4}$$

There are two cases to consider:

- When $\hat{F}(\hat{X}_i) \sqsubseteq \hat{X}_i$ and $\hat{X}_{i+1} = \hat{X}_i$: we have $\hat{F}(\hat{X}_i) \sqsubseteq \hat{X}_{i+1}$ and therefore $\hat{F}^{i+1}(\hat{\bot}) \sqsubseteq \hat{X}_{i+1}$.
- **2** When $\hat{F}(\hat{X}_i) \not\sqsubseteq \hat{X}_i$ and $\hat{X}_{i+1} = \hat{X}_i \bigtriangledown \hat{F}_i(\hat{X}_i)$: by the condition of $\bigtriangledown, \hat{F}(\hat{X}_i) \sqsubseteq \hat{X}_i \bigtriangledown \hat{F}_i(\hat{X}_i) = \hat{X}_{i+1}$. Thus, $\hat{F}(\hat{X}_i) \sqsubseteq \hat{X}_{i+1}$ for all $i \in \mathbb{N}$. By (4), we have $\hat{F}^{i+1}(\hat{\bot}) \sqsubseteq \hat{X}_{i+1}$.

Proof of Narrowing's Safety

• We first show that $\{\hat{F}(\hat{Y}_i)\}_i$ is a decreasing chain (if so, by the second condition of narrowing, the narrowing sequence $\{\hat{Y}_i\}_i$ eventually stabilizes). We can show that $\{\hat{F}(\hat{Y}_i)\}_i$ is a decreasing chain by showing the following

$$\forall i \in \mathbb{N}. \ \hat{Y}_i \sqsupseteq \hat{F}(\hat{Y}_i).$$
(5)

This is because, from (5), we have $\hat{Y}_i \supseteq \hat{Y}_i \triangle \hat{F}(\hat{Y}_i) \supseteq \hat{F}(\hat{Y}_i)$ and by the mono. of \hat{F} , we have

$$\hat{F}(\hat{Y}_i) \sqsupseteq \hat{F}(\hat{Y}_i \bigtriangleup \hat{F}(\hat{Y}_i)) = \hat{F}(\hat{Y}_{i+1})$$

Proof of (5): exercise.

• We next show that $\forall i \in \mathbb{N}$. $\hat{F}^i(\hat{\perp}) \sqsubseteq \hat{Y}_i$ by induction (exercise).