# AAA616: Program Analysis

# Denotational Semantics

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# Denotational Semantics

- In denotational semantics, we are interested in the mathematical meaning of a program.
- Also called compositional semantics: The meaning of an expression is defined with the meanings of its immediate subexpressions.
- Denotational semantics for **While**:

$$\begin{array}{rcl} a & \rightarrow & n \mid x \mid a_1 + a_2 \mid a_1 \star a_2 \mid a_1 - a_2 \\ b & \rightarrow & \texttt{true} \mid \texttt{false} \mid a_1 = a_2 \mid a_1 \leq a_2 \mid \neg b \mid b_1 \wedge b_2 \\ c & \rightarrow & x := a \mid \texttt{skip} \mid c_1 ; c_2 \mid \texttt{if}\ b\ c_1\ c_2 \mid \texttt{while}\ b\ c \end{array}$$

# Denotational Semantics of Expressions

$$
\begin{array}{rcl}
\mathcal{A}[\![a]\!] & : & \textsf{State} \to \mathbb{Z} \\
\mathcal{A}[\![n]\!](s) & = & n \\
\mathcal{A}[\![x]\!](s) & = & s(x) \\
\mathcal{A}[\![a_1 + a_2]\!](s) & = & \mathcal{A}[\![a_1]\!](s) + \mathcal{A}[\![a_2]\!](s) \\
\mathcal{A}[\![a_1 \star a_2]\!](s) & = & \mathcal{A}[\![a_1]\!](s) \times \mathcal{A}[\![a_2]\!](s) \\
\mathcal{A}[\![a_1 - a_2]\!](s) & = & \mathcal{A}[\![a_1]\!](s) - \mathcal{A}[\![a_2]\!](s)
\end{array}
$$

$$
\begin{array}{rcl}
\mathcal{B}[\![b]\!] & : & \textsf{State} \to \textsf{T} \\
\mathcal{B}[\![\texttt{true}]\!](s) & = & \mathit{true} \\
\mathcal{B}[\![\texttt{false}]\!](s) & = & \mathit{false} \\
\mathcal{B}[\![a_1 = a_2]\!](s) & = & \mathcal{A}[\![a_1]\!](s) = \mathcal{A}[\![a_2]\!](s) \\
\mathcal{B}[\![a_1 \leq a_2]\!](s) & = & \mathcal{A}[\![a_1]\!](s) \leq \mathcal{A}[\![a_2]\!](s) \\
\mathcal{B}[\![\neg b]\!](s) & = & \mathcal{B}[\![b]\!](s) = \mathit{false} \\
\mathcal{B}[\![b_1 \wedge b_2]\!](s) & = & \mathcal{B}[\![b_1]\!](s) \wedge \mathcal{B}[\![b_2]\!](s)
\end{array}
$$

# Denotational Semantics of Commands

$$
\begin{array}{rcl}
\mathcal{C}[\![c]\!] & : & \textbf{State} \hookrightarrow \textbf{State} \\
\mathcal{C}[\![x := a]\!](s) & = & s[x \mapsto \mathcal{A}[\![a]\!](s)] \\
\mathcal{C}[\![\texttt{skip}]\!] & = & \textbf{id} \\
\mathcal{C}[\![c_1; c_2]\!] & = & \mathcal{C}[\![c_2]\!] \circ \mathcal{C}[\![c_1]\!] \\
\mathcal{C}[\![\texttt{if}\ b\ c_1\ c_2]\!] & = & \textbf{cond}(\mathcal{B}[\![b]\!], \mathcal{C}[\![c_1]\!], \mathcal{C}[\![c_2]\!]) \\
\mathcal{C}[\![\texttt{while}\ b\ c]\!] & = & \mathit{fix} F
\end{array}
$$

where

$$
\textbf{cond}(f, g, h) = \lambda s. \begin{cases} g(s) & \cdots f(s) = \mathit{true} \\ h(s) & \cdots f(s) = \mathit{false} \end{cases}
$$

$$
F(g) = \textbf{cond}(\mathcal{B}[\![b]\!], g \circ \mathcal{C}[\![c]\!], \textbf{id})
$$

# Denotational Semantics of Loops

The meaning of the while loop is the mathematical object (i.e. partial function in **State** $\hookrightarrow$ **State**) that satisfies the equation:

$$\mathcal{C}[\![\texttt{while}\ b\ c]\!] = \textbf{cond}(\mathcal{B}[\![b]\!], \mathcal{C}[\![\texttt{while}\ b\ c]\!] \circ \mathcal{C}[\![c]\!], \textbf{id}).$$

Rewrite the equation:

$$\mathcal{C}[\![\texttt{while}\ b\ c]\!] = F(\mathcal{C}[\![\texttt{while}\ b\ c]\!])$$

where

$$F(g) = \textbf{cond}(\mathcal{B}[\![b]\!], g \circ \mathcal{C}[\![c]\!], \textbf{id}).$$

The meaning of the while loop is defined as the least fixed point of $F$:

$$\mathcal{C}[\![\texttt{while}\ b\ c]\!] = \mathit{fix} F$$

where $\mathit{fix} F$ denotes the *least fixed point* of $F$.

# Example

$$\texttt{while } \neg(x = 0) \texttt{ skip}$$

- $F$
- $\mathit{fix}\, F$

## Questions

- Does the least fixed point $\mathit{fix}F$ exist?
- Is $\mathit{fix}F$ unique?
- How to compute $\mathit{fix}F$?

# Fixed Point Theory

**Theorem**

*Let $f : D \to D$ be a continuous function on a CPO $D$. Then $f$ has a (unique) least fixed point, $\mathit{fix}(f)$, and*

$$\mathit{fix}(f) = \bigsqcup_{n \geq 0} f^n(\bot).$$

The denotational semantics is well-defined if

- **State** $\hookrightarrow$ **State** is a CPO, and
- $F$ : (**State** $\hookrightarrow$ **State**) $\to$ (**State** $\hookrightarrow$ **State**) is a continuous function.

# Plan

- Complete Partial Order
- Continuous Functions
- Least Fixed Point

# Partially Ordered Set

## Definition (Partial Order)

We say a binary relation $\sqsubseteq$ is a partial order on a set $D$ iff $\sqsubseteq$ is

- reflexive: $\forall p \in D.\ p \sqsubseteq p$
- transitive: $\forall p, q, r \in D.\ p \sqsubseteq q \ \wedge\ q \sqsubseteq r \implies p \sqsubseteq r$
- anti-symmetric: $\forall p, q \in D.\ p \sqsubseteq q \ \wedge\ q \sqsubseteq p \implies p = q$

We call such a pair $(D, \sqsubseteq)$ partially ordered set, or poset.

## Lemma

*If a partially ordered set $(D, \sqsubseteq)$ has a least element $d$, then $d$ is unique.*

# Exercise 1

Let $S$ be a non-empty set. Prove that $(\wp(S), \subseteq)$ is a partially ordered set.

# Exercise 2

Let $X \hookrightarrow Y$ be the set of all partial functions from a set $X$ to a set $Y$, and define $f \sqsubseteq g$ iff

$$\mathbf{dom}(f) \subseteq \mathbf{dom}(g) \ \wedge \ \forall x \in \mathbf{dom}(f).\ f(x) = g(x).$$

Prove that $(X \hookrightarrow Y, \sqsubseteq)$ is a partially ordered set.

# Least Upper Bound

## Definition (Least Upper Bound)

Let $(D, \sqsubseteq)$ be a partially ordered set and let $Y$ be a subset of $D$. An upper bound of $Y$ is an element $d$ of $D$ such that

$$\forall d' \in Y.\ d' \sqsubseteq d.$$

An upper bound $d$ of $Y$ is a least upper bound if and only if $d \sqsubseteq d'$ for every upper bound $d'$ of $Y$. The least upper bound of $Y$ is denoted by $\bigsqcup Y$. The least upper bound (lub, join) of $a$ and $b$ is written as $a \sqcup b$.

## Lemma

*If $Y$ has a least upper bound $d$, then $d$ is unique.*

# Greatest Lower Bound

## Definition (Greatest Lower Bound)

Let $(D, \sqsubseteq)$ be a partially ordered set and let $Y$ be a subset of $D$. A lower bound of $Y$ is an element $d$ of $D$ such that

$$\forall d' \in Y.\ d \sqsubseteq d'.$$

An lower bound $d$ of $Y$ is a greatest lower bound if and only if $d' \sqsubseteq d$ for every lower bound $d'$ of $Y$. The greatest lower bound of $Y$ is denoted by $\bigsqcap Y$. The greatest lower bound (glb, meet) of $a$ and $b$ is written as $a \sqcap b$.

# Chain

## Definition (Chain)

Let $(D, \sqsubseteq)$ be a poset and $Y$ a subset of $D$. $Y$ is called a chain if $Y$ is totally ordered:

$$\forall y_1, y_2 \in Y. y_1 \sqsubseteq y_2 \text{ or } y_2 \sqsubseteq y_1.$$

## Example

Consider the poset $(\wp(\{a, b, c\}), \subseteq)$.

- $Y_1 = \{\emptyset, \{a\}, \{a, c\}\}$
- $Y_2 = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$

# Complete Partial Order (CPO)

**Definition (CPO)**

A poset $(D, \sqsubseteq)$ is a CPO, if every chain $Y \subseteq D$ has $\bigsqcup Y \in D$.

**Lemma**

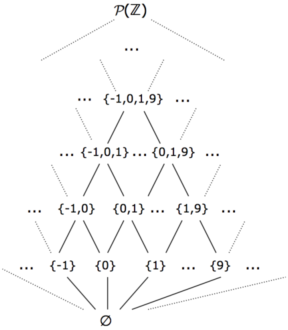
*If $(D, \sqsubseteq)$ is a CPO, then it has a least element $\perp$ given by $\perp = \bigsqcup \emptyset$.*

* We denote the least element and the greatest element in a poset as $\perp$ and $\top$, respectively, if they exist.

# Examples

## Example

Let $S$ be a non-empty set. Then, $(\wp(S), \subseteq)$ is a CPO. The lub $\bigsqcup Y$ for $Y$ is $\bigcup Y$. The least element is $\emptyset$.



# Examples

## Example

The poset $(X \hookrightarrow Y, \sqsubseteq)$ of all partial functions from a set $X$ to a set $Y$, equipped with the partial order

$$\textbf{dom}(f) \subseteq \textbf{dom}(g) \ \wedge \ \forall x \in \textbf{dom}(f).\ f(x) = g(x)$$

is a CPO (but not a complete lattice). The lub of a chain $Y$ is the partial function $f$ with $\textbf{dom}(f) = \bigcup_{f_i \in Y} \textbf{dom}(f_i)$ and

$$f(x) = \begin{cases} f_n(x) & \cdots x \in \textbf{dom}(f_i) \text{ for some } f_i \in Y \\ \textbf{undef} & \cdots \textit{otherwise} \end{cases}$$

The least element $\bot = \lambda x.\textbf{undef}$.

# Lattices

Ordered sets with richer structures.

### Definition (Lattice)

A lattice $(D, \sqsubseteq, \sqcup, \sqcap)$ is a poset where the lub and glb always exist:

$$\forall a, b \in D.\ a \sqcup b \in D \land a \sqcap b \in D.$$

### Definition (Complete Lattice)

A complete lattice $(D, \sqsubseteq, \sqcup, \sqcap, \bot, \top)$ is a poset such that every subset $Y \subseteq D$ has $\bigsqcup Y \in D$ and $\bigsqcap Y \in D$, and $D$ has a least element $\bot = \bigsqcup \emptyset = \bigsqcap D$ and a greatest element $\top = \bigsqcap \emptyset = \bigsqcup D$.

* A complete lattice is a CPO.

# Derived Ordered Structures

When $(D_1, \sqsubseteq_1, \sqcup_1, \sqcap_1, \bot_1, \top_1)$ and $(D_2, \sqsubseteq_2, \sqcup_2, \sqcap_2, \bot_2, \top_2)$ are complete lattices (resp., CPO), so are the following ordered sets:

- Lifting: $(D_1 \cup \{\bot\}, \sqsubseteq, \sqcup, \sqcap, \bot, \top)$
  - $\bot \not\in D_1$ is a new element
  - $a \sqsubseteq b \iff a = \bot \vee a \sqsubseteq_1 b$
  - $\bot \sqcup a = a \sqcup \bot = a$ and otherwise $a \sqcup b = a \sqcup_1 b$ (similar for $\sqcap$)
  - $\top = \top_1$
- Cartesian product: $(D_1 \times D_2, \sqsubseteq, \sqcup, \sqcap, \bot, \top)$.
- Pointwise lifting: $(S \to D, \sqsubseteq, \sqcup, \sqcap, \bot, \top)$ ($S$ is a set)
  - $a \sqsubseteq b \iff \forall s \in S.\ a(s) \sqsubseteq_1 b(s)$
  - $\forall s \in S.\ (a \sqcup b)(s) \iff a(s) \sqcup_1 b(s)$
  - $\forall s \in S.\ \bot(s) = \bot_1$

# Monotone Functions

## Definition (Monotone Functions)

A function $f : D \to E$ between posets is *monotone* iff

$$\forall d, d' \in D.\ d \sqsubseteq d' \implies f(d) \sqsubseteq f(d').$$

# Example

Consider $(\wp(\{a,b,c\}), \subseteq)$ and $(\wp(\{d,e\}), \subseteq)$ and two functions $f_1, f_2 : \wp(\{a,b,c\}) \rightarrow \wp(\{d,e\})$

| $X$ | $\{a,b,c\}$ | $\{a,b\}$ | $\{a,c\}$ | $\{b,c\}$ | $\{a\}$ | $\{b\}$ | $\{c\}$ | $\emptyset$ |
|---|---|---|---|---|---|---|---|---|
| $f_1(X)$ | $\{d,e\}$ | $\{d\}$ | $\{d,e\}$ | $\{d,e\}$ | $\{d\}$ | $\{d\}$ | $\{e\}$ | $\emptyset$ |

| $X$ | $\{a,b,c\}$ | $\{a,b\}$ | $\{a,c\}$ | $\{b,c\}$ | $\{a\}$ | $\{b\}$ | $\{c\}$ | $\emptyset$ |
|---|---|---|---|---|---|---|---|---|
| $f_2(X)$ | $\{d\}$ | $\{d\}$ | $\{d\}$ | $\{e\}$ | $\{d\}$ | $\{e\}$ | $\{e\}$ | $\{e\}$ |

# Exercise

Determine which of the following functionals of

$$(\mathbf{State} \hookrightarrow \mathbf{State}) \to (\mathbf{State} \hookrightarrow \mathbf{State})$$

are monotone:

1. $F_0(g) = g$.
2. $F_1(g) = \begin{cases} g_1 & \cdots g = g_2 \\ g_2 & \cdots \textit{otherwise} \end{cases}$ where $g_1 \neq g_2$.
3. $F_2(g) = \lambda s. \begin{cases} g(s) & \cdots s(x) \neq 0 \\ s & \cdots s(x) = 0 \end{cases}$

# Properties of Monotone Functions

**Lemma**

*Let* $(D_1, \sqsubseteq_1)$, $(D_2, \sqsubseteq_2)$, *and* $(D_3, \sqsubseteq_3)$ *be CPOs. Let* $f : D_1 \to D_2$ *and* $g : D_2 \to D_3$ *be monotone functions. Then,* $g \circ f : D_1 \to D_3$ *is a monotone function.*

# Properties of Monotone Functions

**Lemma**

*Let* $(D_1, \sqsubseteq_1)$ *and* $(D_2, \sqsubseteq_2)$ *be CPOs. Let* $f : D_1 \to D_2$ *be a monotone function. If* $Y$ *is a chain in* $D_1$*, then* $f(Y) = \{f(d) \mid d \in Y\}$ *is a chain in* $D_2$*. Furthermore,*

$$\bigsqcup f(Y) \sqsubseteq f(\bigsqcup Y).$$

# Continuous Functions

## Definition (Continuous Functions)

A function $f : D_1 \to D_2$ defined on CPOs $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ is continuous if it is monotone and it preserves least upper bounds of chains:

$$\bigsqcup f(Y) = f(\bigsqcup Y)$$

for all non-empty chains $Y$ in $D_1$. If $f(\bigsqcup Y) = \bigsqcup f(Y)$ holds for the empty chain (that is, $\bot = f(\bot)$), then we say that $f$ is strict.

# Properties of Continuous Functions

**Lemma**

*Let* $f : D_1 \to D_2$ *be a monotone function defined on posets* $(D_1, \sqsubseteq_1)$ *and* $(D_2, \sqsubseteq_2)$ *and* $D_1$ *is a finite set. Then,* $f$ *is continuous.*

# Properties of Continuous Functions

## Lemma

*Let* $(D_1, \sqsubseteq_1)$, $(D_2, \sqsubseteq_2)$, *and* $(D_3, \sqsubseteq_3)$ *be CPOs. Let* $f : D_1 \to D_2$ *and* $g : D_2 \to D_3$ *be continuous functions. Then,* $g \circ f : D_1 \to D_3$ *is a continuous function.*

# Least Fixed Points

## Definition (Fixed Point)

Let $(D, \sqsubseteq)$ be a poset. A *fixed point* of a function $f : D \to D$ is an element $d \in D$ such that $f(d) = d$. We write $\mathit{fix}(f)$ for the *least fixed point* of $f$, if it exists, such that

- $f(\mathit{fix}(f)) = \mathit{fix}(f)$
- $\forall d \in D.\ f(d) = d \implies \mathit{fix}(f) \sqsubseteq d$

* More notations:

- $x$ is a fixed point of $f$ if $f(x) = x$. Let $\mathbf{fp}(f) = \{x \mid f(x) = x\}$ be the set of fixed points.
- $x$ is a pre-fixed point of $f$ if $x \sqsubseteq f(x)$.
- $x$ is a post-fixed point of $f$ if $x \sqsupseteq f(x)$.
- $\mathbf{lfp}(f)$: the least fixed point
- $\mathbf{gfp}(f)$: the greatest fixed point

# Fixed Point Theorem

## Theorem (Kleene Fixed Point)

*Let $f : D \to D$ be a continuous function on a CPO $D$. Then $f$ has a least fixed point, $\mathit{fix}(f)$, and*

$$\mathit{fix}(f) = \bigsqcup_{n \geq 0} f^n(\bot)$$

*where* $f^n(\bot) = \begin{cases} \bot & n = 0 \\ f(f^{n-1}(\bot)) & n > 0 \end{cases}$

# Proof

We show the claims of the theorem by showing that $\bigsqcup_{n\geq 0} f^n(\bot)$ exists and it is indeed equivalent to $\mathit{fix}(f)$. First note that $\bigsqcup_{n\geq 0} f^n(\bot)$ exists because $f^0(\bot) \sqsubseteq f^1(\bot) \sqsubseteq f^2(\bot) \sqsubseteq \ldots$ is a chain. We show by induction that $\forall n \in \mathbb{N}. f^n(\bot) \sqsubseteq f^{n+1}(\bot)$:

- $\bot \sqsubseteq f(\bot)$ ($\bot$ is the least element)
- $f^n(\bot) \sqsubseteq f^{n+1}(\bot) \implies f^{n+1}(\bot) \sqsubseteq f^{n+2}(\bot)$ (monotonicity of $f$)

Now, we show that $\mathit{fix}(f) = \bigsqcup_{n\geq 0} f^n(\bot)$ in two steps:

- We show that $\bigsqcup_{n\geq 0} f^n(\bot)$ is a fixed point of $f$:

$$\begin{aligned} f(\bigsqcup_{n\geq 0} f^n(\bot)) &= \bigsqcup_{n\geq 0} f(f^n(\bot)) && \text{continuity of } f \\ &= \bigsqcup_{n\geq 0} f^{n+1}(\bot) \\ &= \bigsqcup_{n\geq 0} f^n(\bot) \end{aligned}$$

# Proofs

- We show that $\bigsqcup_{n\geq 0} f^n(\bot)$ is smaller than all the other fixed points. Suppose $d$ is a fixed point, i.e., $f(d) = d$. Then,

$$\bigsqcup_{n\geq 0} f^n(\bot) \sqsubseteq d$$

since $\forall n \in \mathbb{N}. f^n(\bot) \sqsubseteq d$:

$$f^0(\bot) = \bot \sqsubseteq d, \qquad f^n(\bot) \sqsubseteq d \implies f^{n+1}(\bot) \sqsubseteq f(d) = d.$$

Therefore, we conclude

$$\mathit{fix}(f) = \bigsqcup_{n\geq 0} f^n(\bot).$$

# Well-definedness of the Semantics

The function $F$

$$F(g) = \textbf{cond}(\mathcal{B}[\![b]\!], g \circ \mathcal{C}[\![c]\!], \textbf{id})$$

is continuous.

**Lemma**

*Let* $g_0 : \textbf{State} \hookrightarrow \textbf{State}, p : \textbf{State} \to \textbf{T}$, *and define*

$$F(g) = \textbf{cond}(p, g, g_0).$$

*Then,* $F$ *is continuous.*

**Lemma**

*Let* $g_0 : \textbf{State} \hookrightarrow \textbf{State}$, *and define*

$$F(g) = g \circ g_0.$$

*Then* $F$ *is continuous.*