AAA528: Computational Logic

Lecture 11 — Static Analysis Examples

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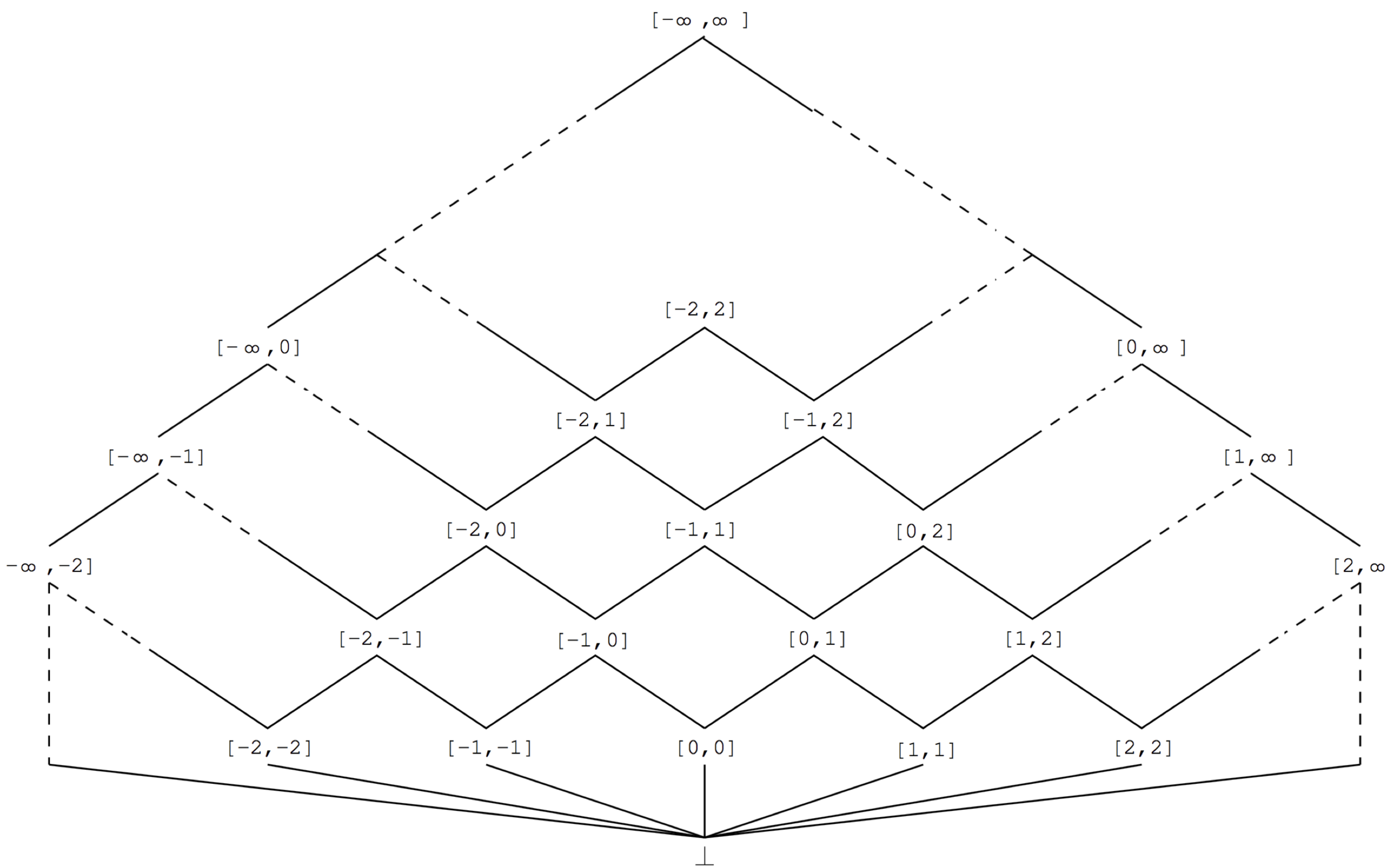
2025 Spring

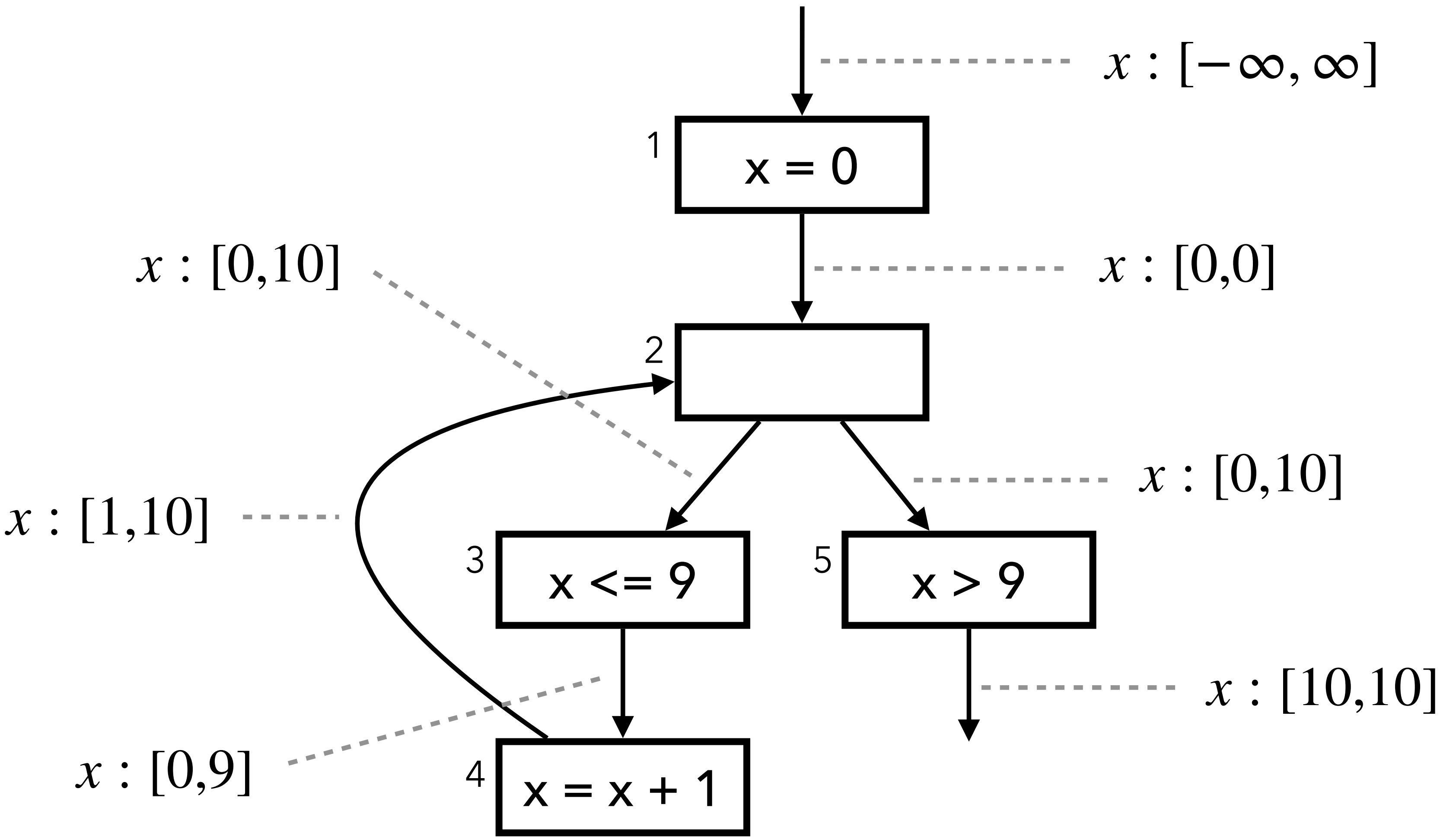
The Interval Domain



Example Program

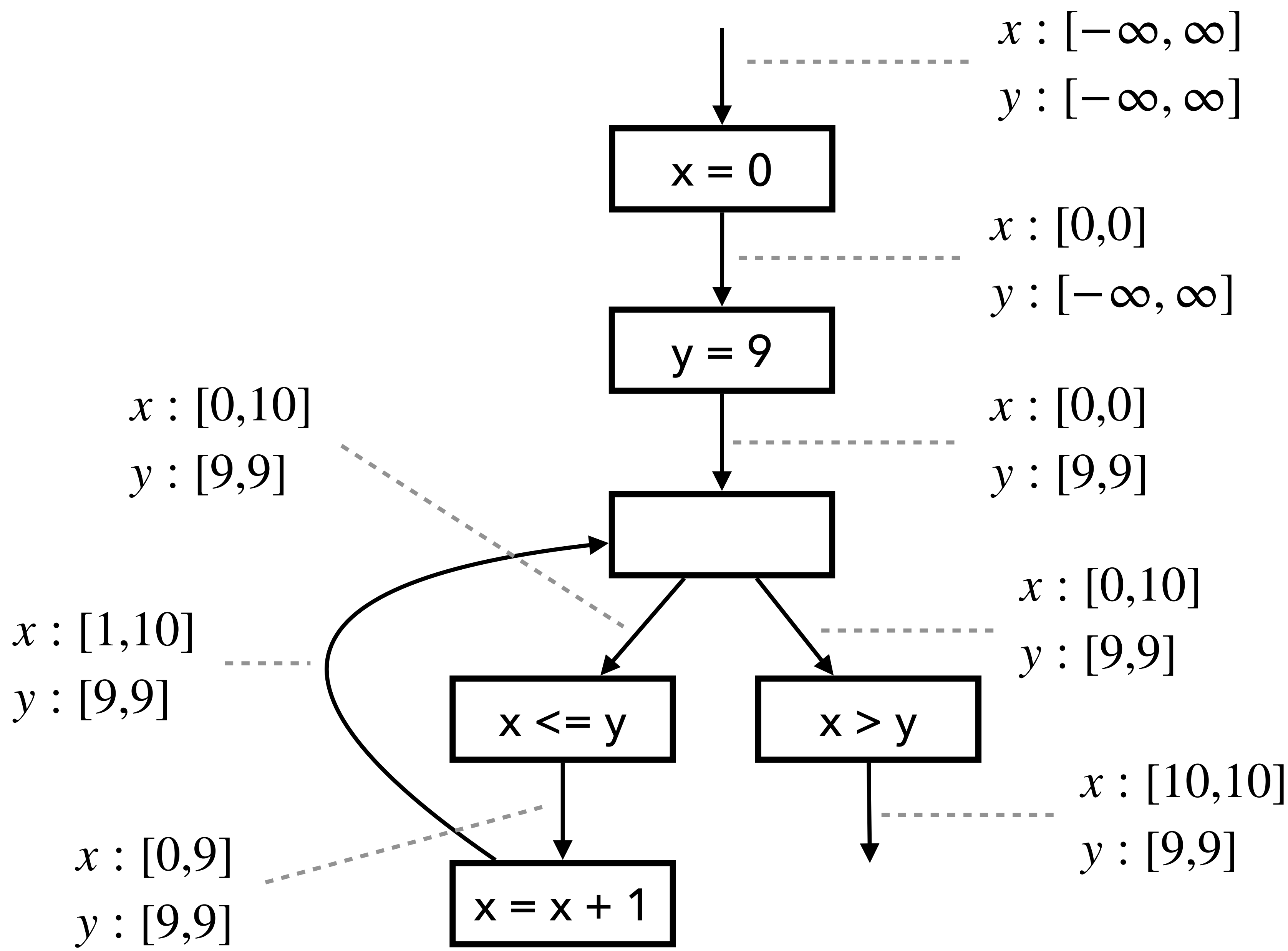
```
x = 0;

while (x <= 9)
  x = x + 1;
```

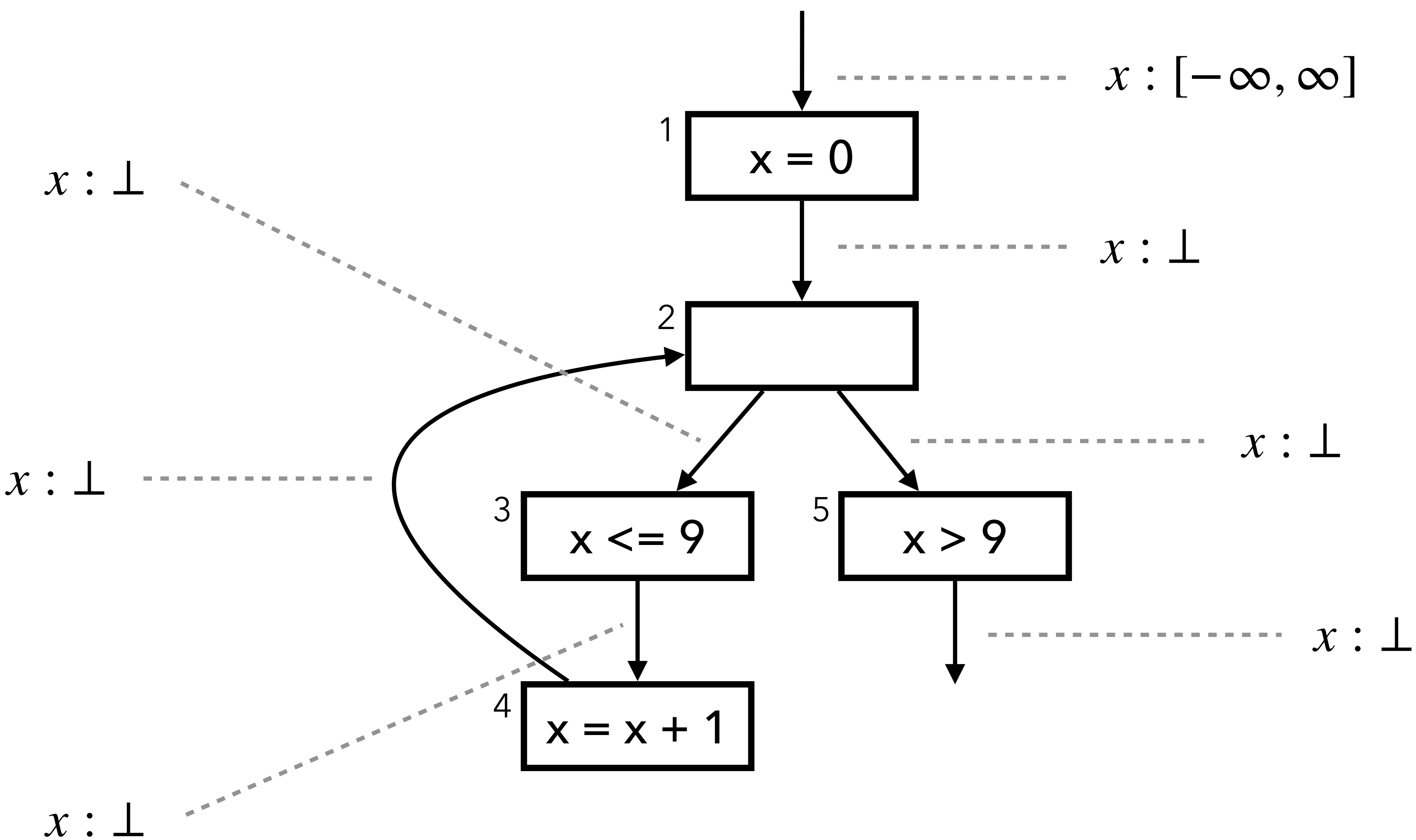


cf. Multiple Variables

```
x = 0;
y = 9
while (x <= y)
  x = x + 1;
```

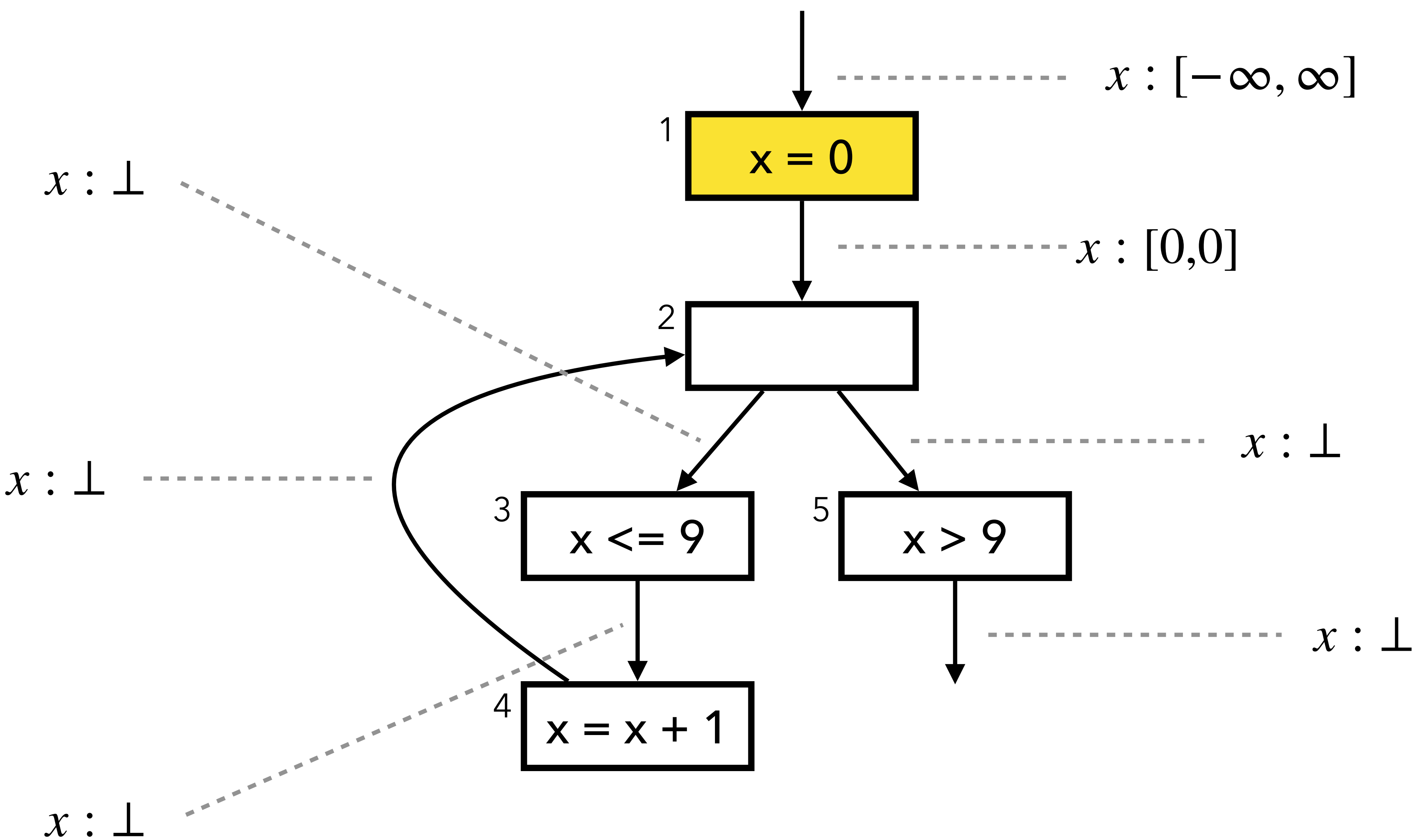


Fixed Point Computation

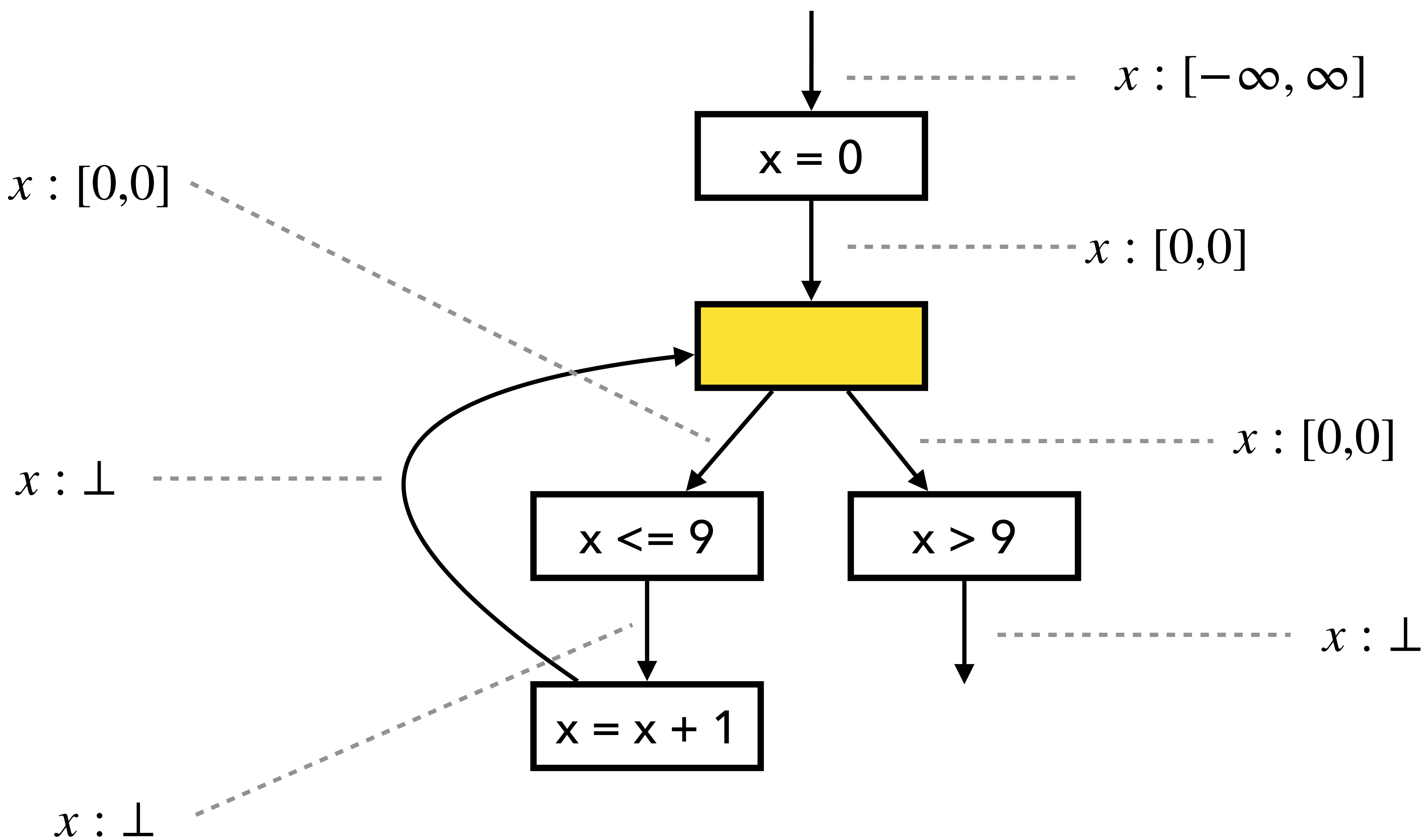


Initial states

Fixed Point Computation

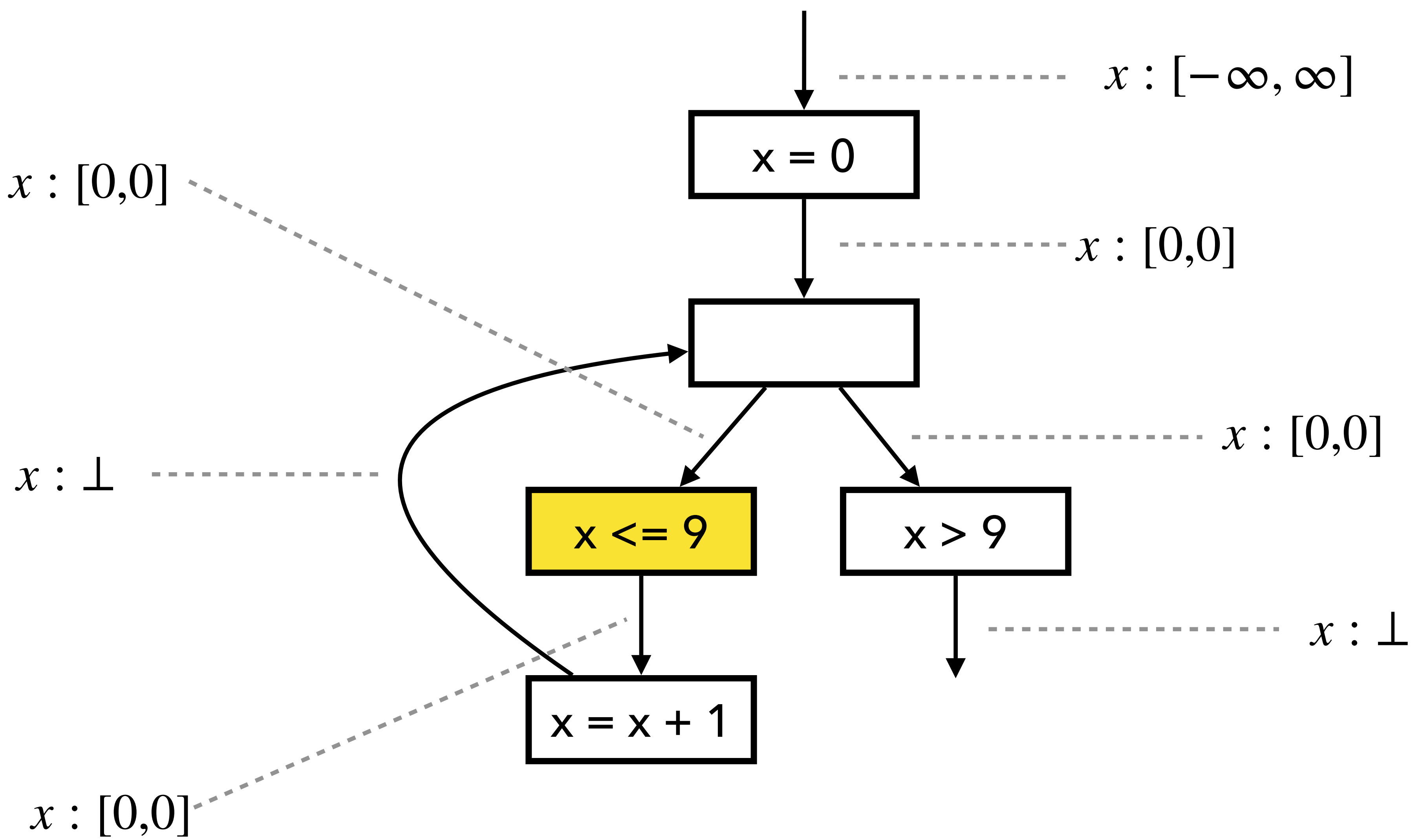


Fixed Point Computation



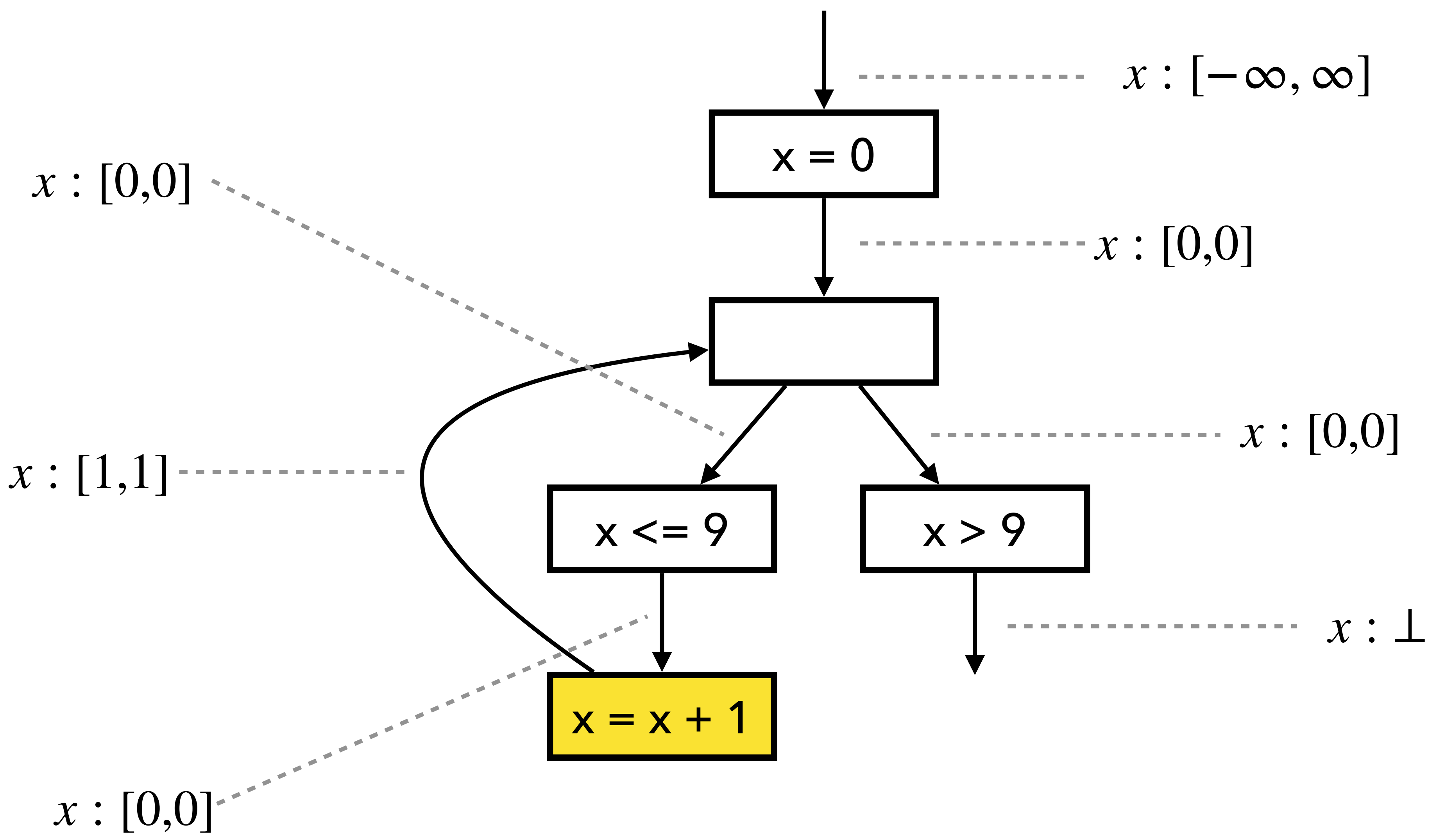
Input state: $[0,0] \sqcup \bot = [0,0]$

Fixed Point Computation

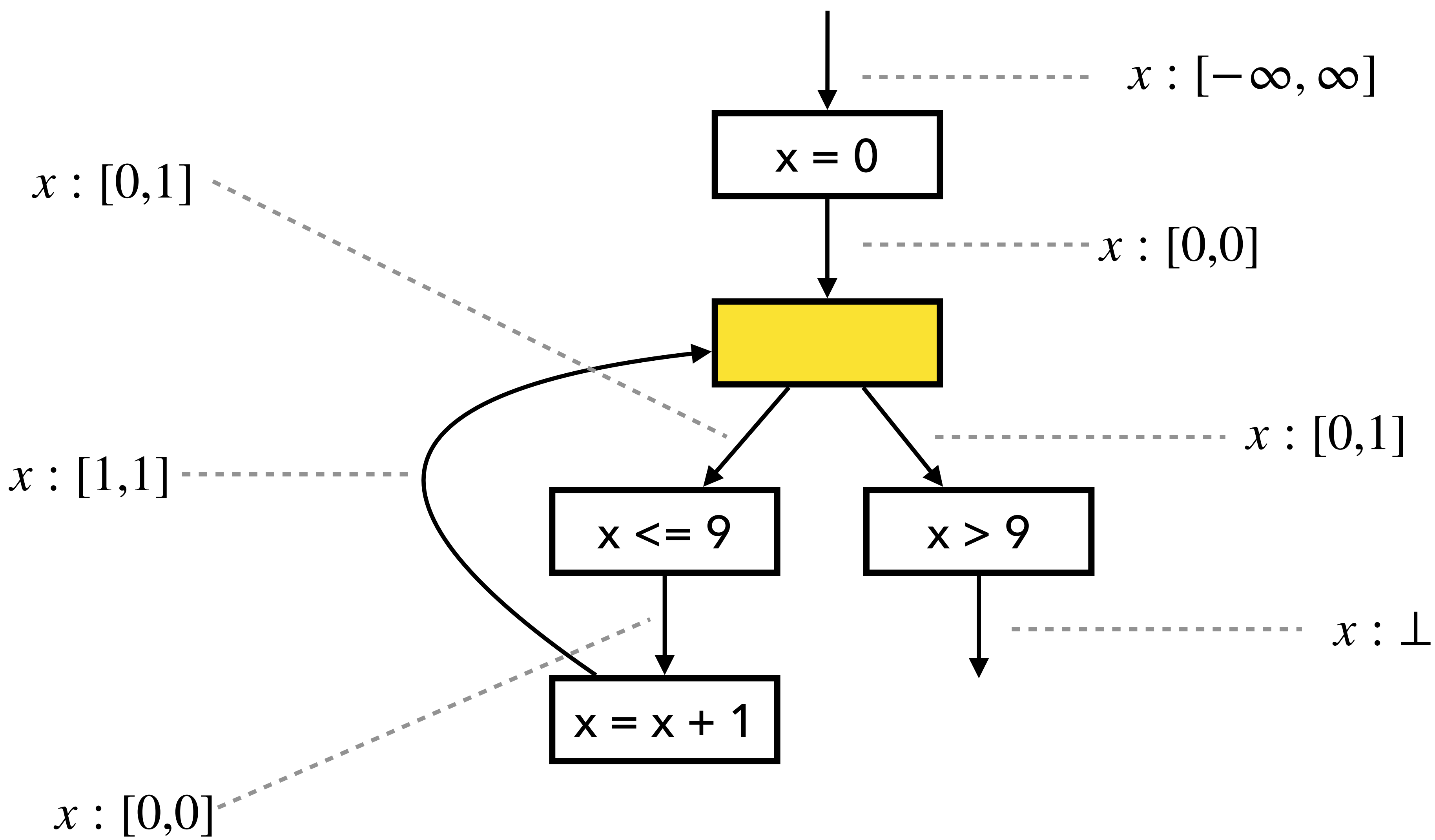


$$[0,0] \sqcap [-\infty,9] = [0,0]$$

Fixed Point Computation

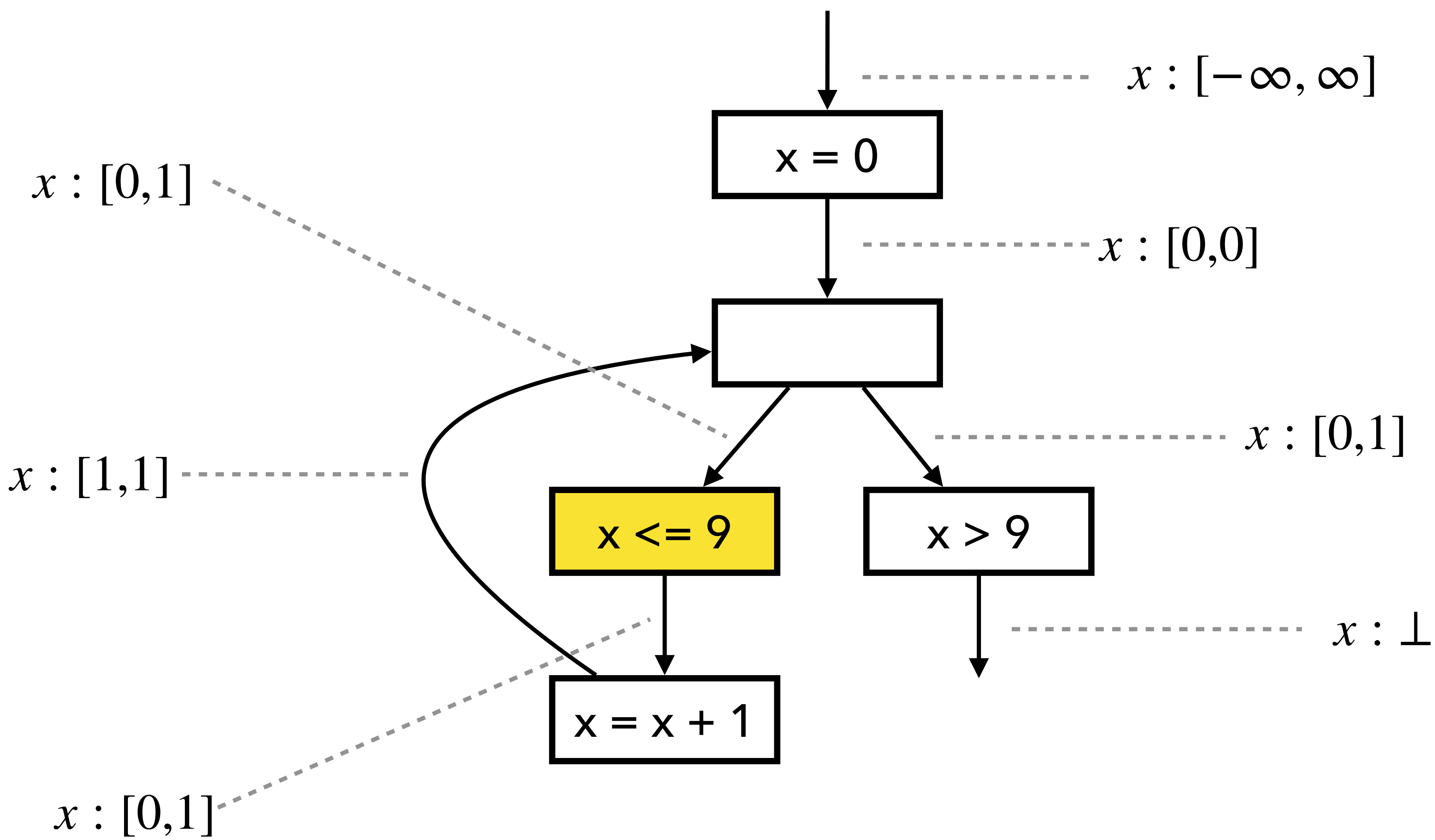


Fixed Point Computation



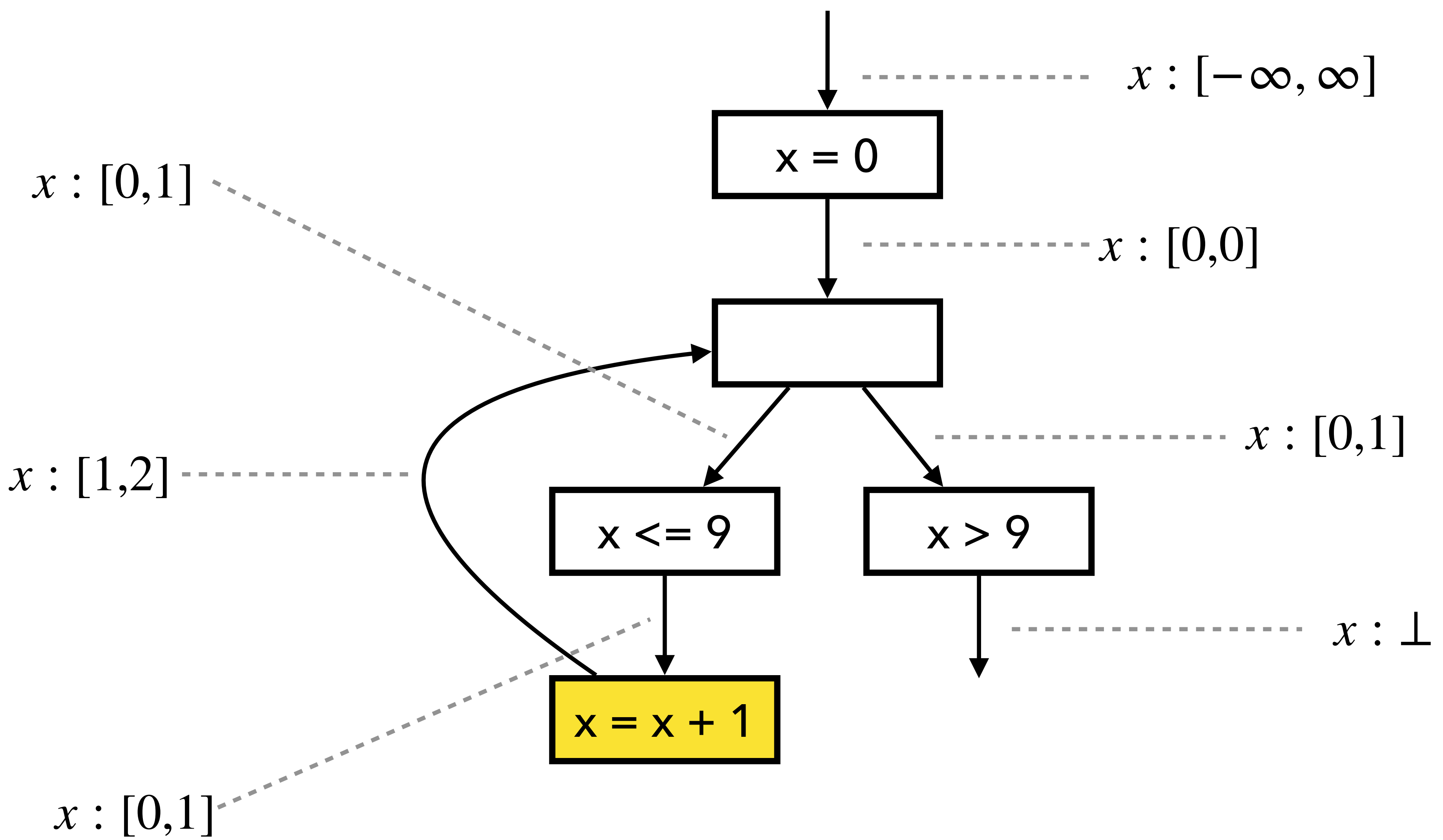
Input state: $[0,0] \sqcup [1,1] = [0,1]$
(1st iteration of loop)

Fixed Point Computation

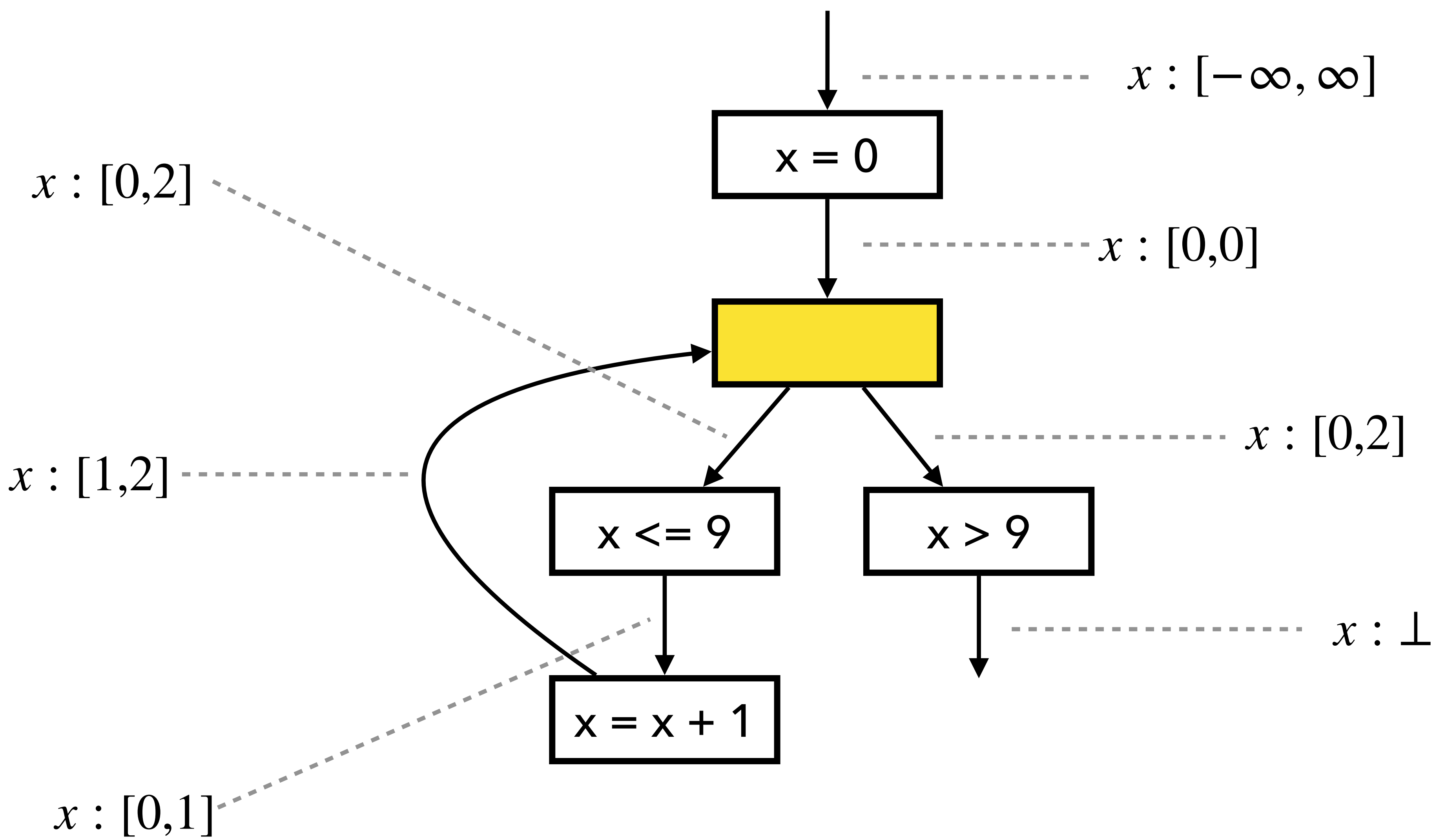


$$[0,1] \sqcap [-\infty,9] = [0,1]$$

Fixed Point Computation

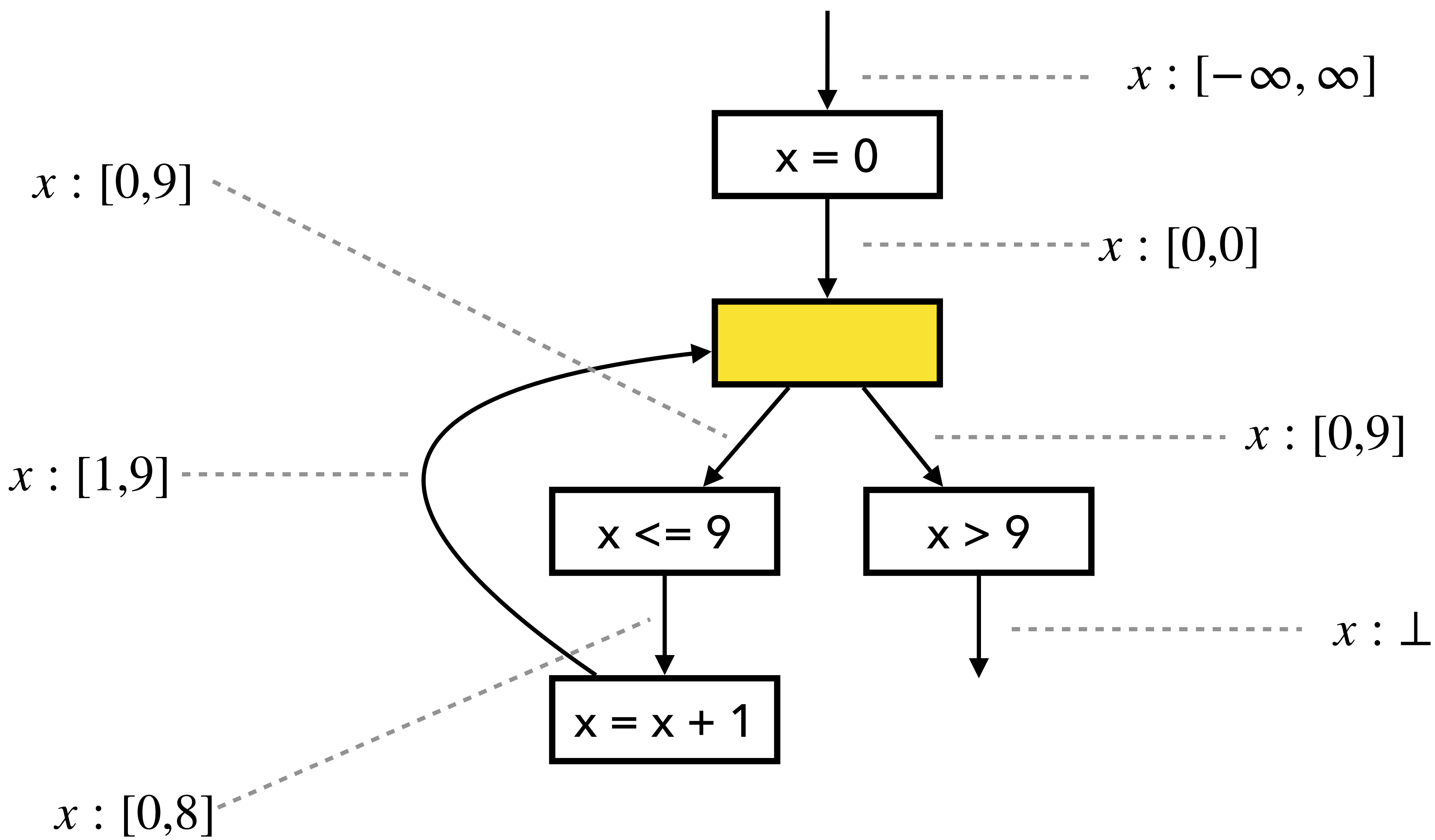


Fixed Point Computation



Input state: $[0,0] \sqcup [1,2] = [0,2]$
(2nd iteration of loop)

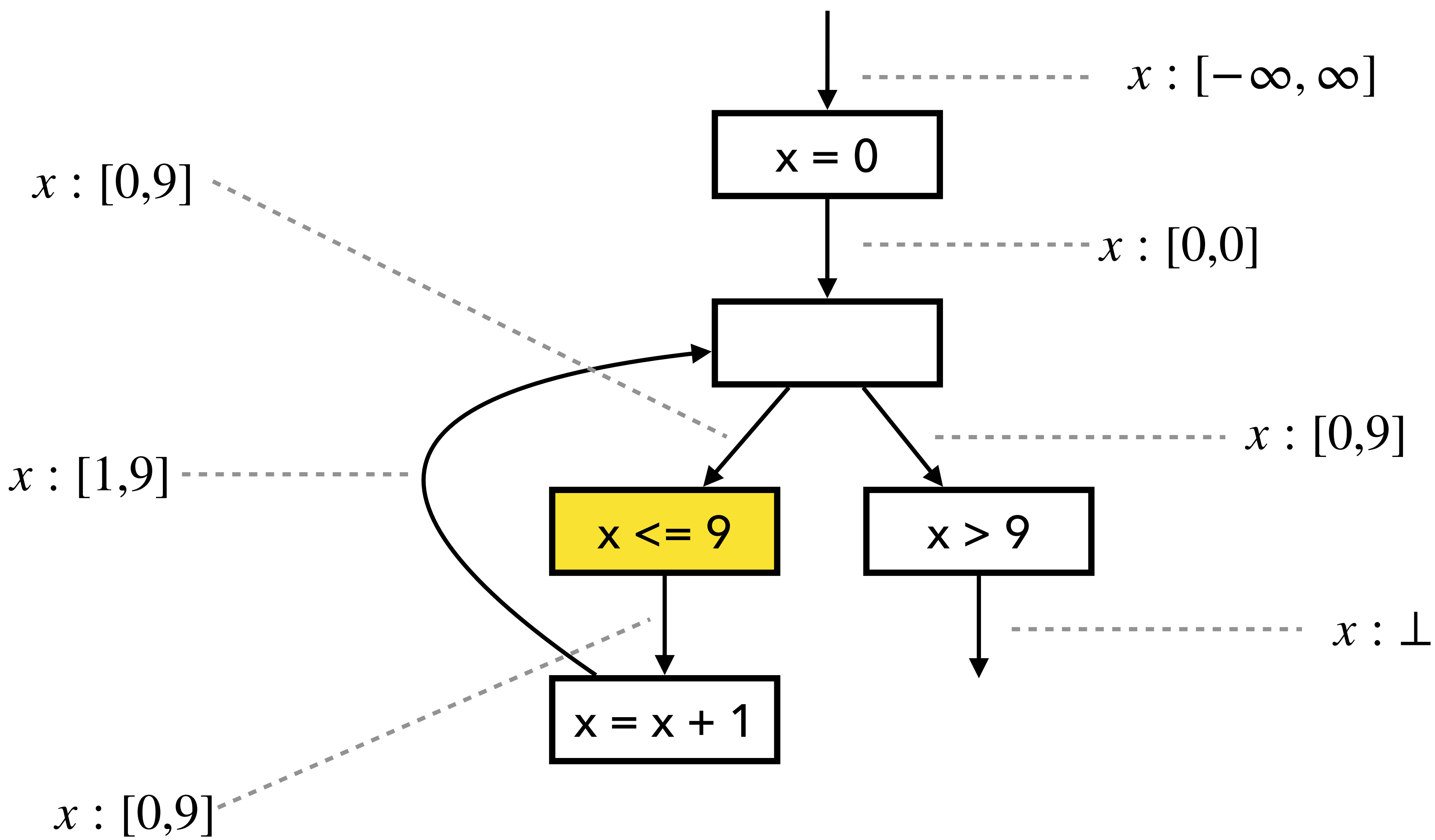
Fixed Point Computation



Input state: $[0,0] \sqcup [1,9] = [0,9]$

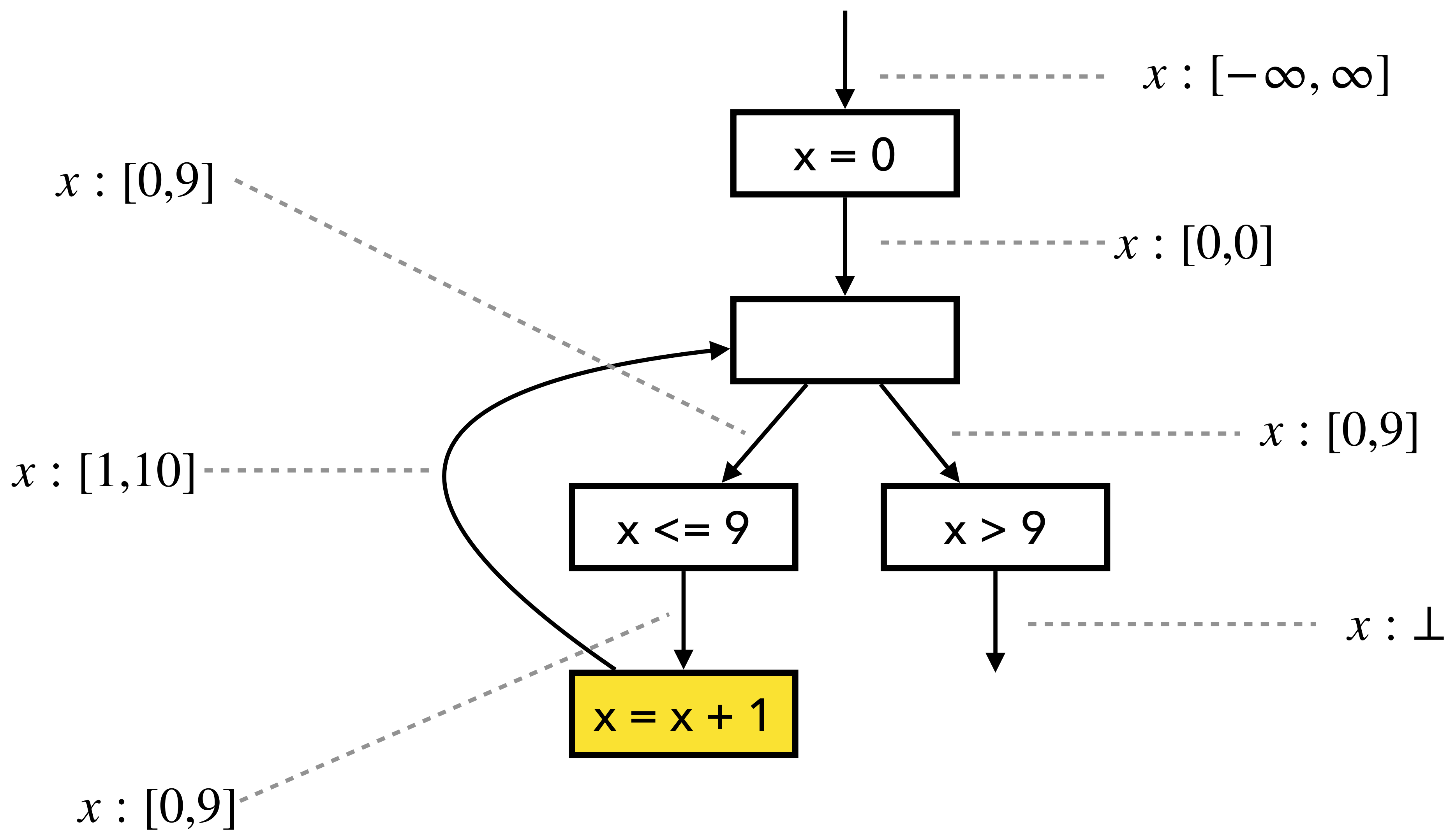
(9th iteration of loop)

Fixed Point Computation

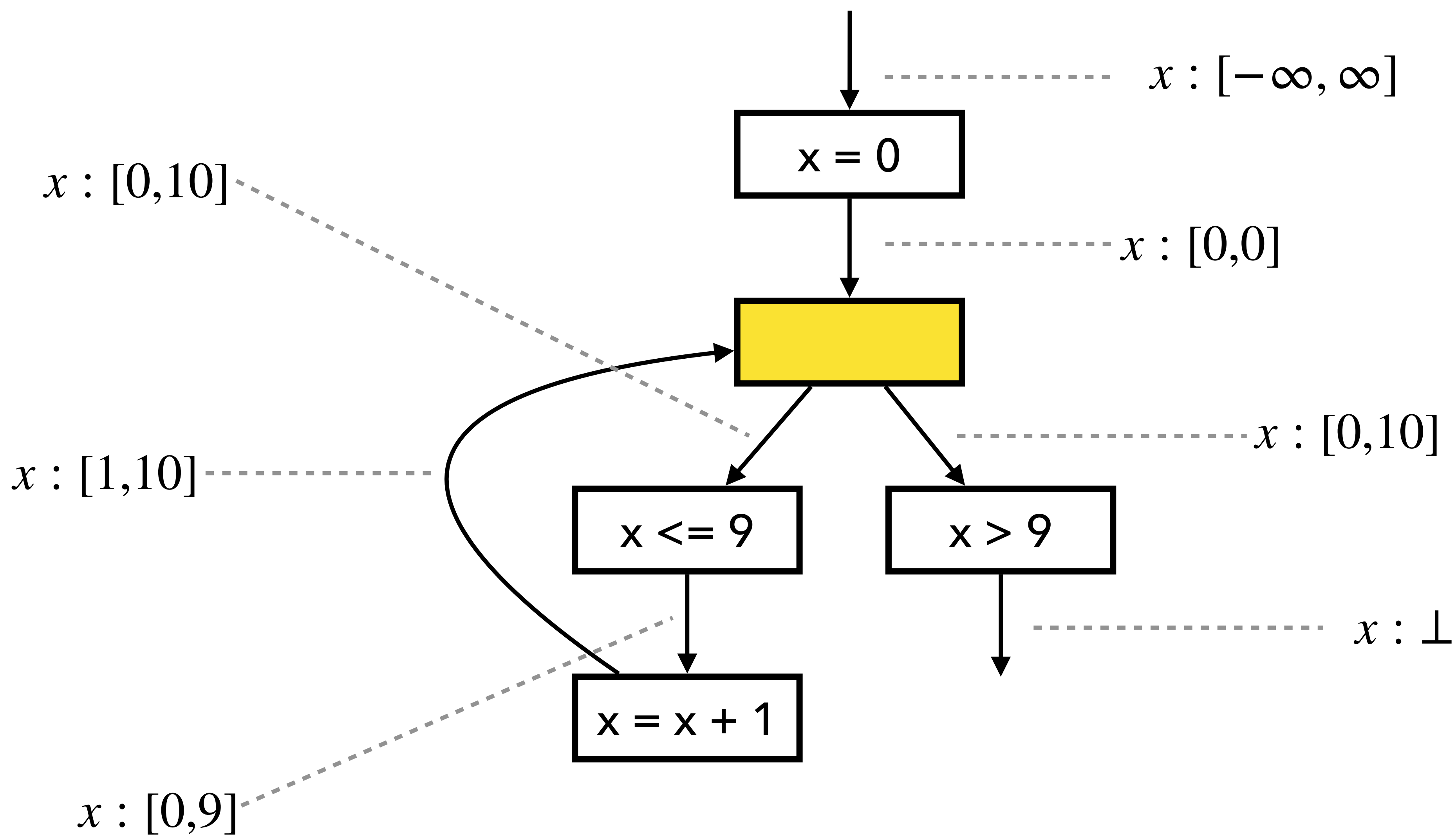


$$[0,9] \sqcap [-\infty,9] = [0,9]$$

Fixed Point Computation



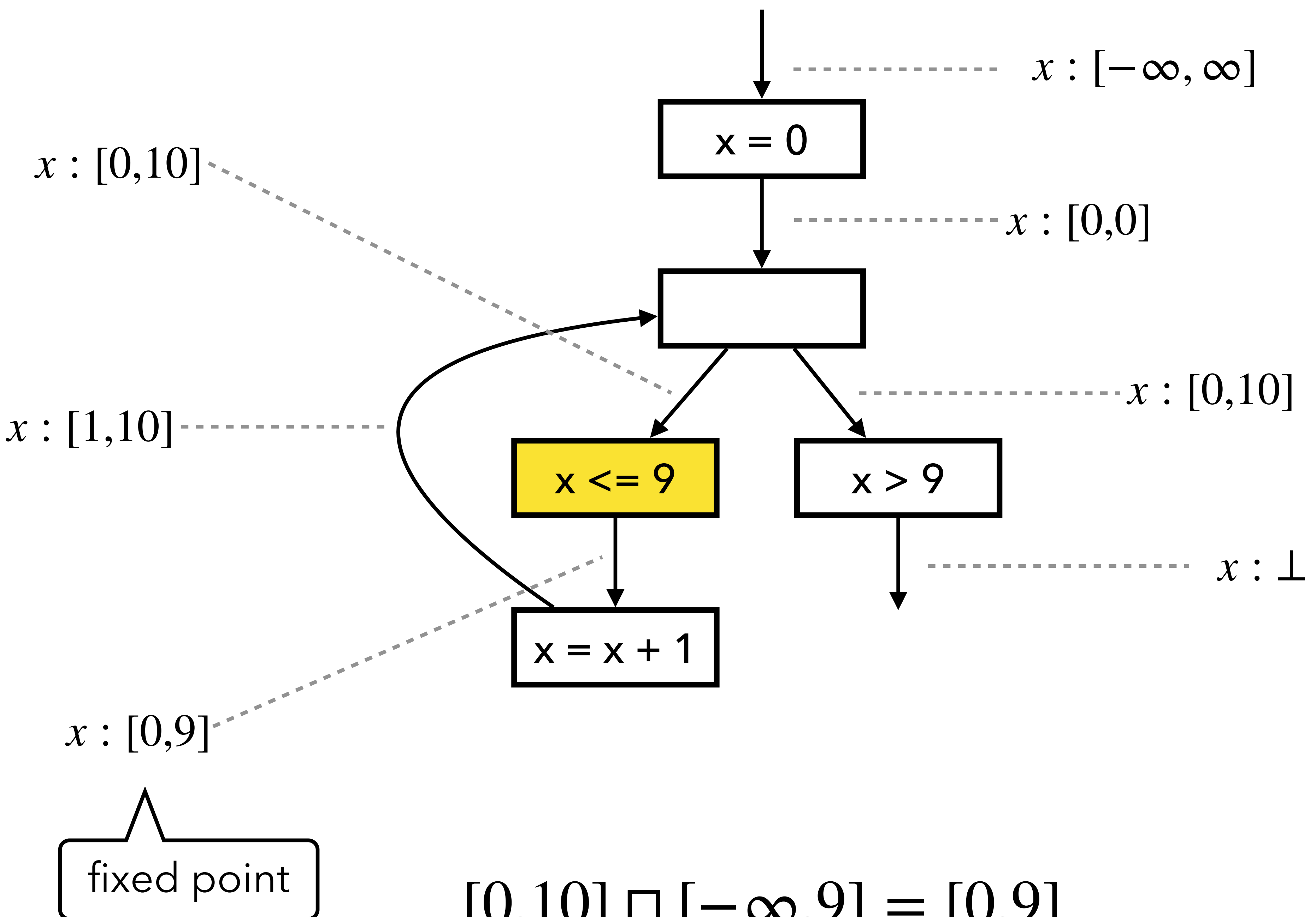
Fixed Point Computation



Input state: $[0,0] \sqcup [1,10] = [0,10]$

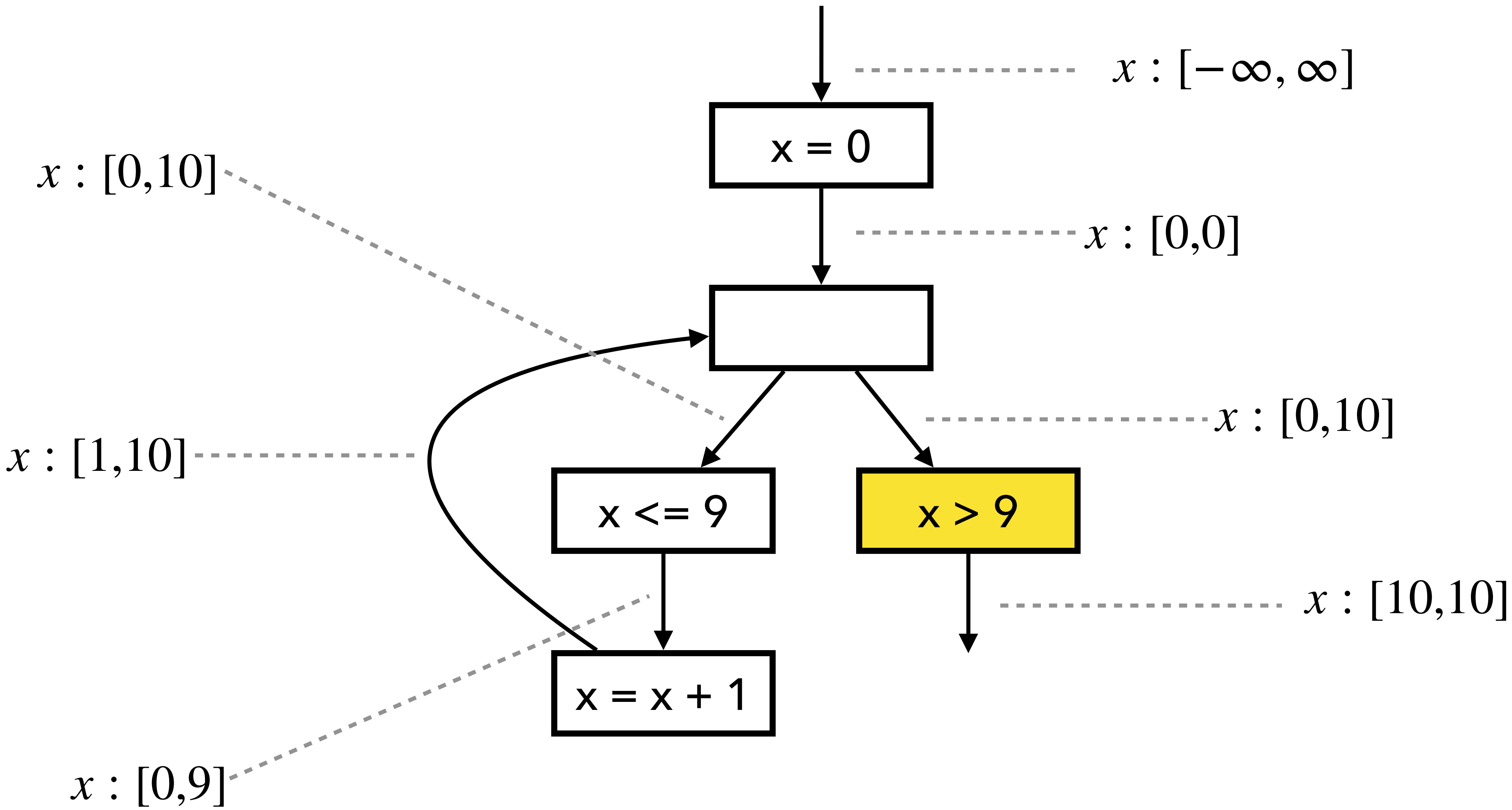
(10th iteration of loop)

Fixed Point Computation



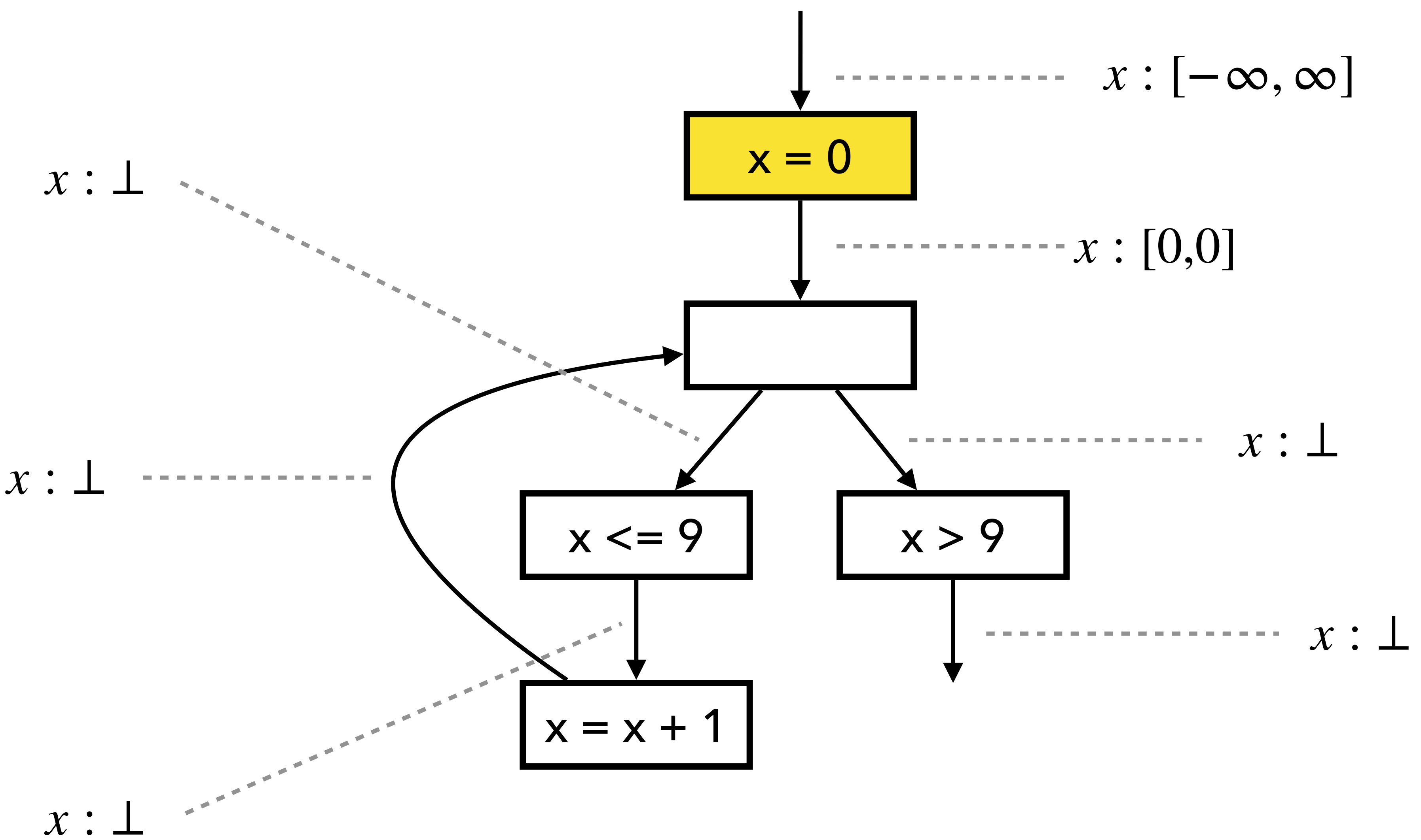
$$[0,10] \sqcap [-\infty,9] = [0,9]$$

Fixed Point Computation

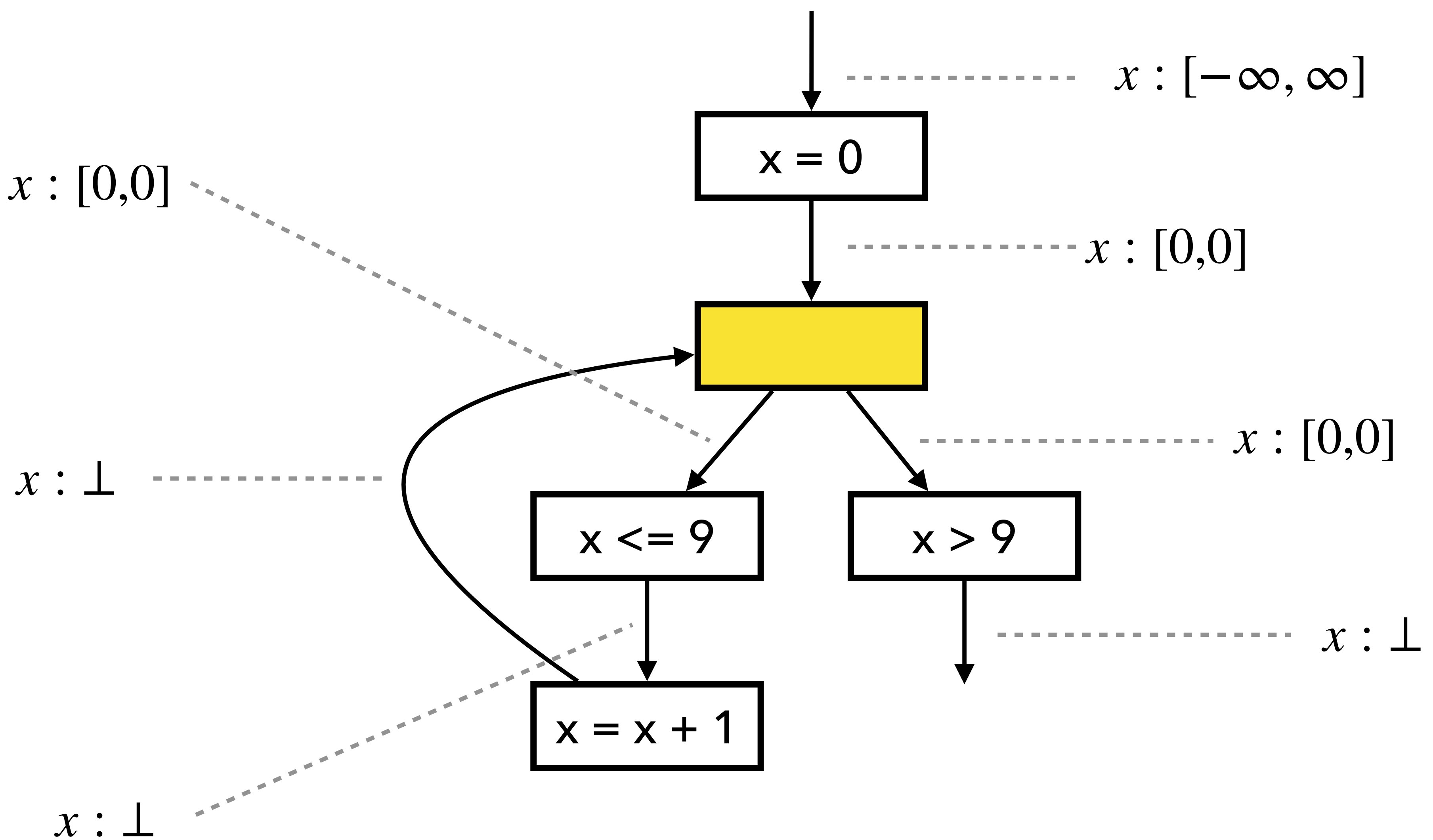


$$[0,10] \sqcap [10,\infty] = [10,10]$$

Fixed Point Comp. with Widening

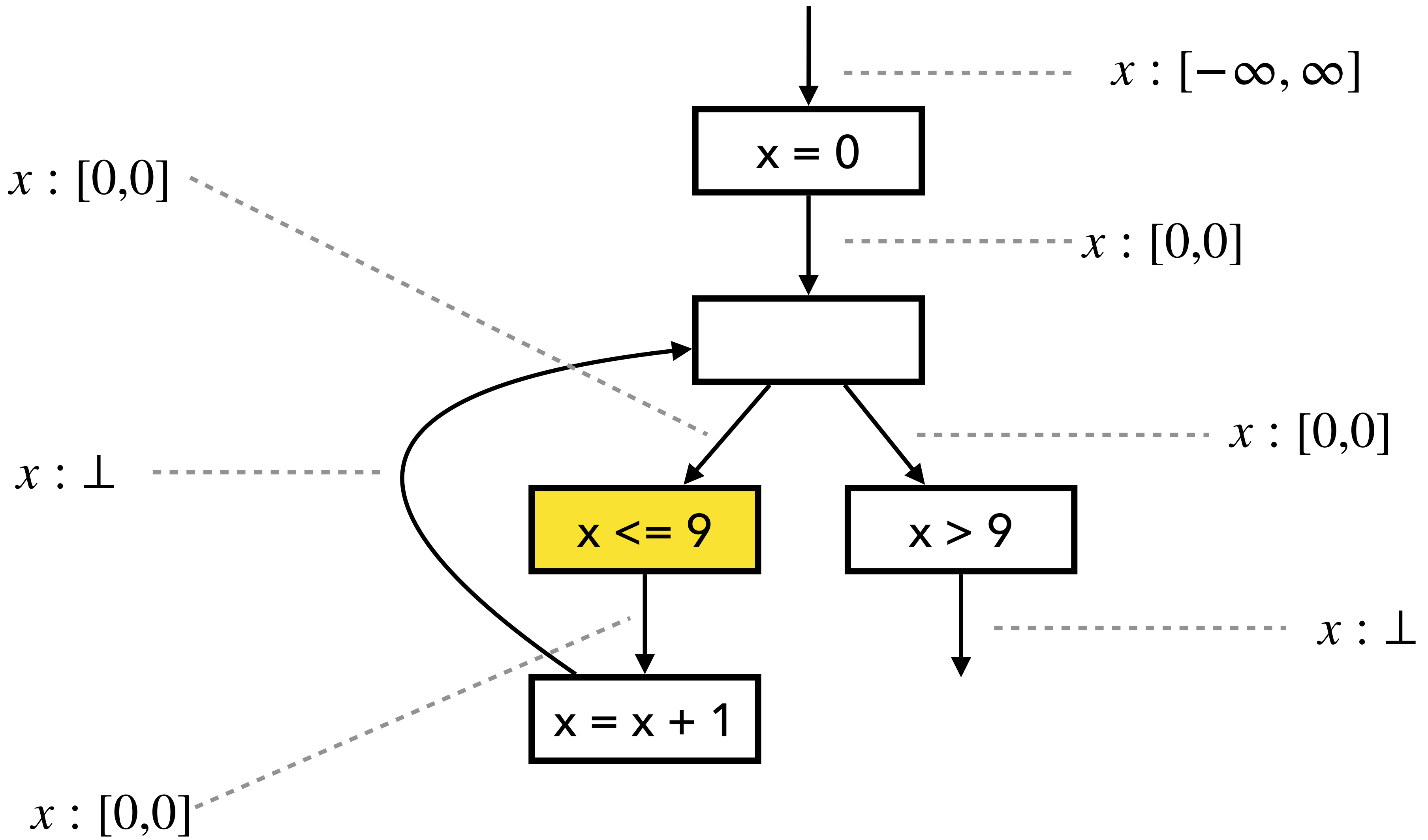


Fixed Point Comp. with Widening



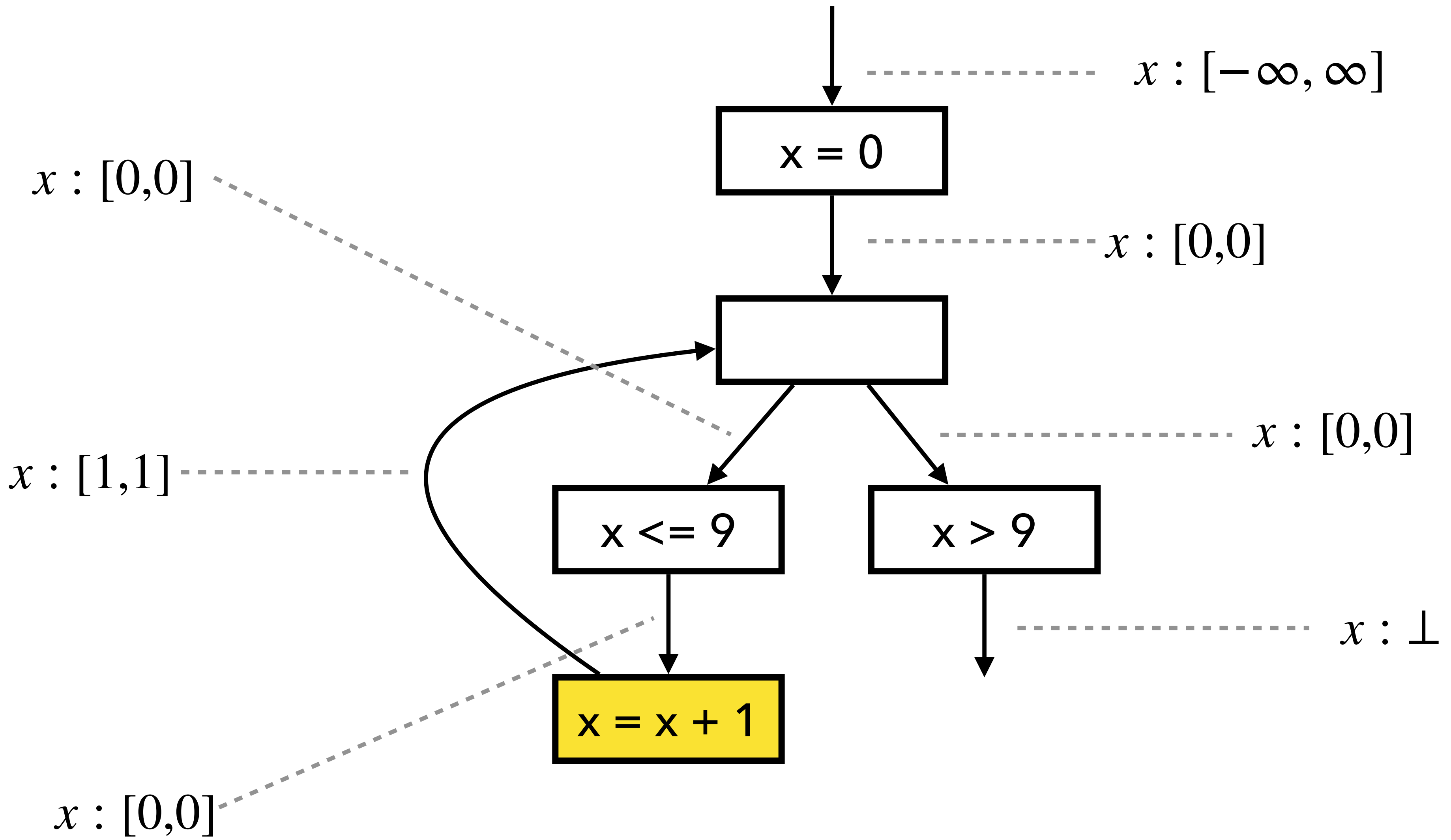
Input state: $[0,0] \sqcup \bot = [0,0]$

Fixed Point Comp. with Widening



$$[0,0] \sqcap [-\infty,9] = [0,0]$$

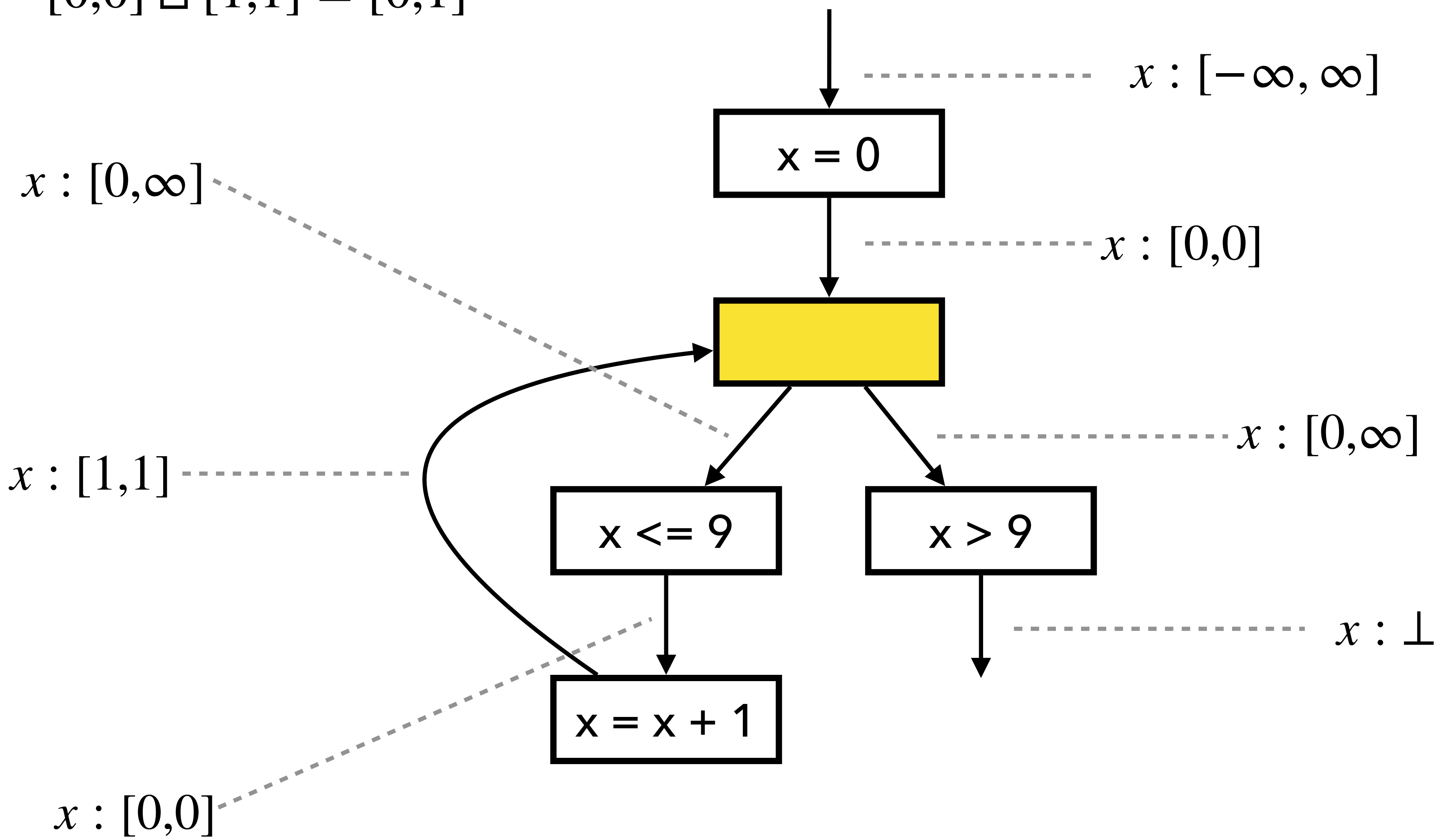
Fixed Point Comp. with Widening



Fixed Point Comp. with Widening

1. Compute output by joining inputs:

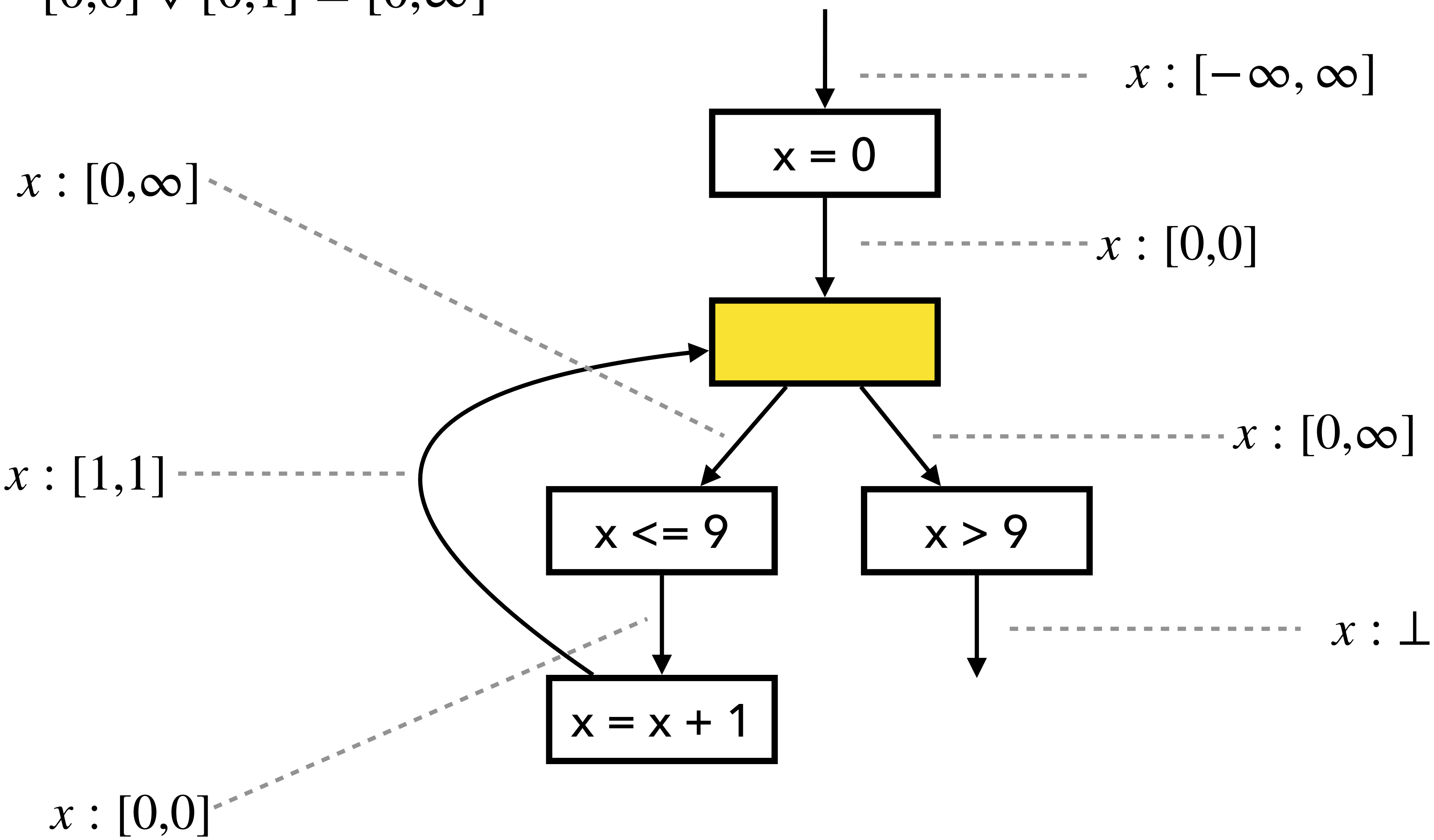
$[0,0] \sqcup [1,1] = [0,1]$



Fixed Point Comp. with Widening

2. Apply widening with old output:

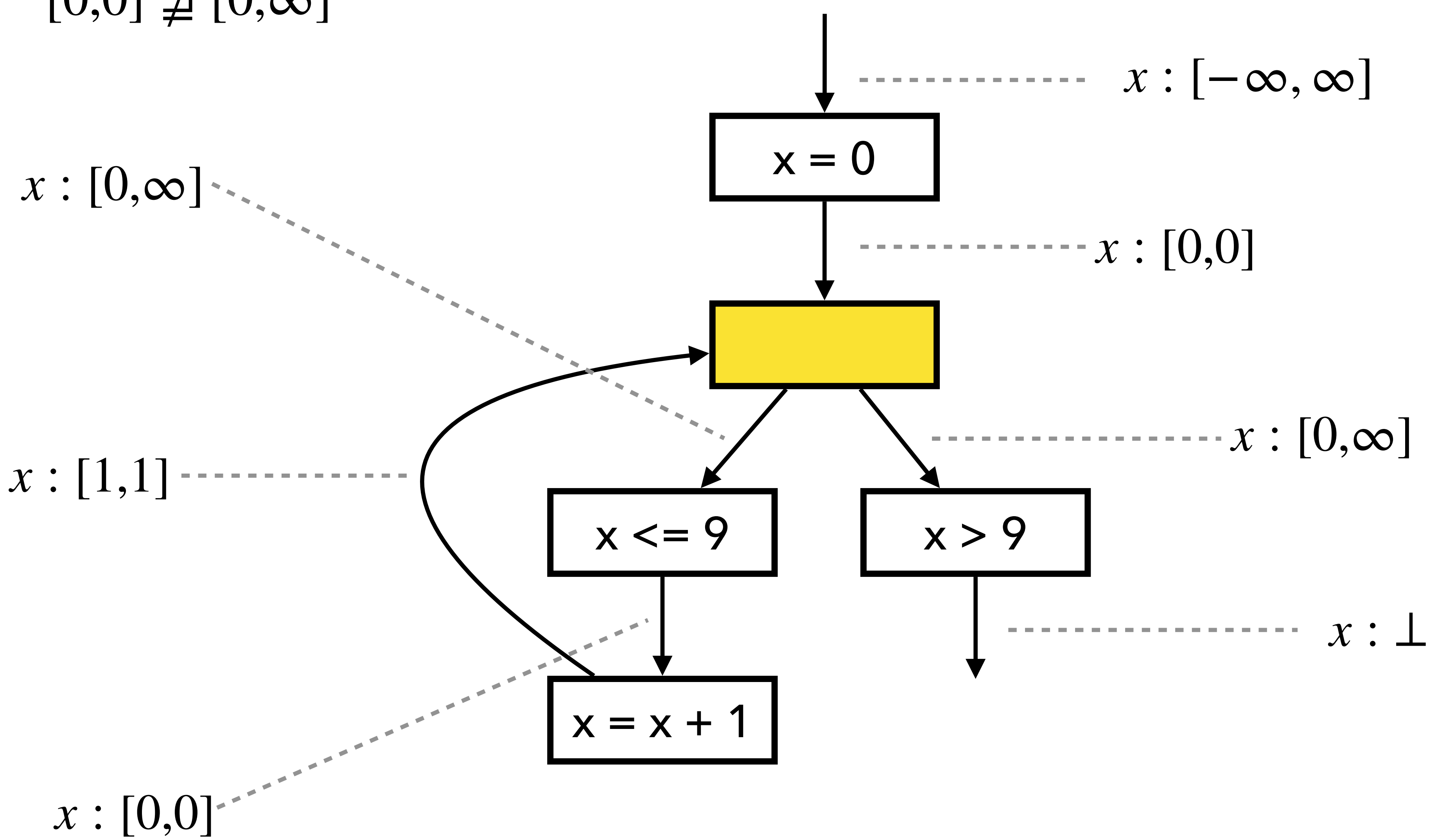
$$[0,0] \triangledown [0,1] = [0,\infty]$$



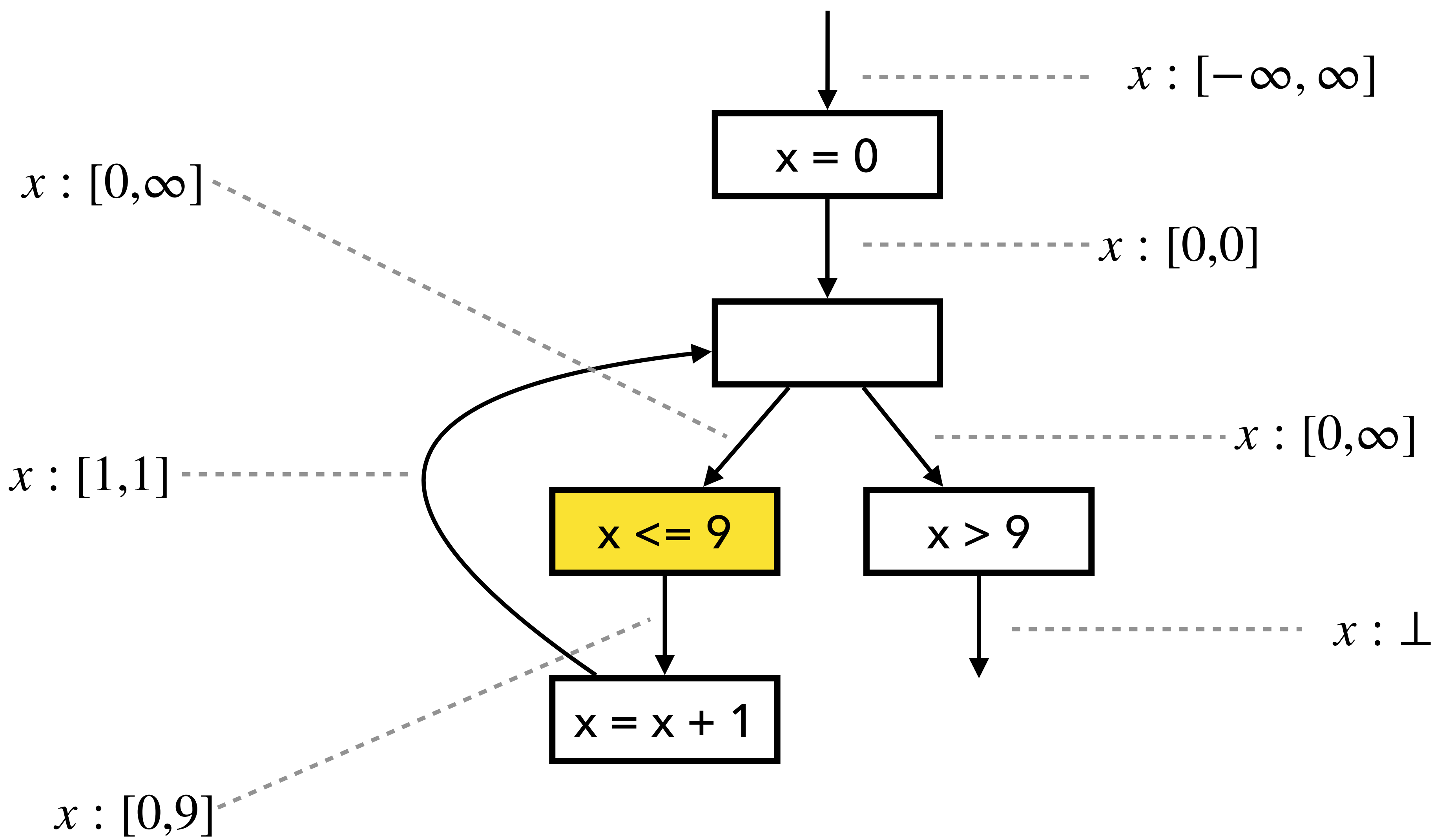
Fixed Point Comp. with Widening

3. Check if fixed point is reached

$$[0,0] \not\sqsupseteq [0,\infty]$$

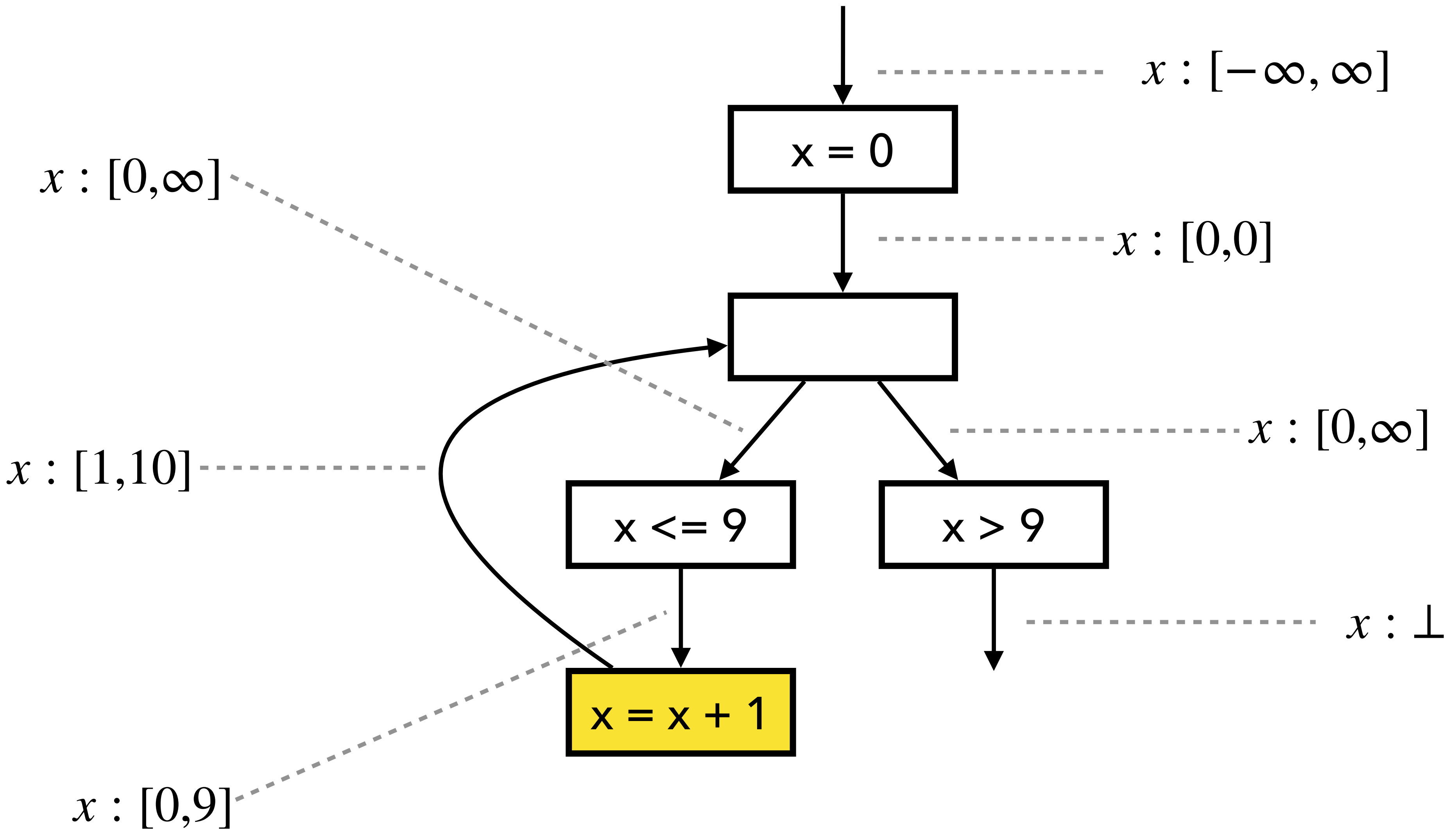


Fixed Point Comp. with Widening



$$[0,\infty] \sqcap [-\infty,9] = [0,9]$$

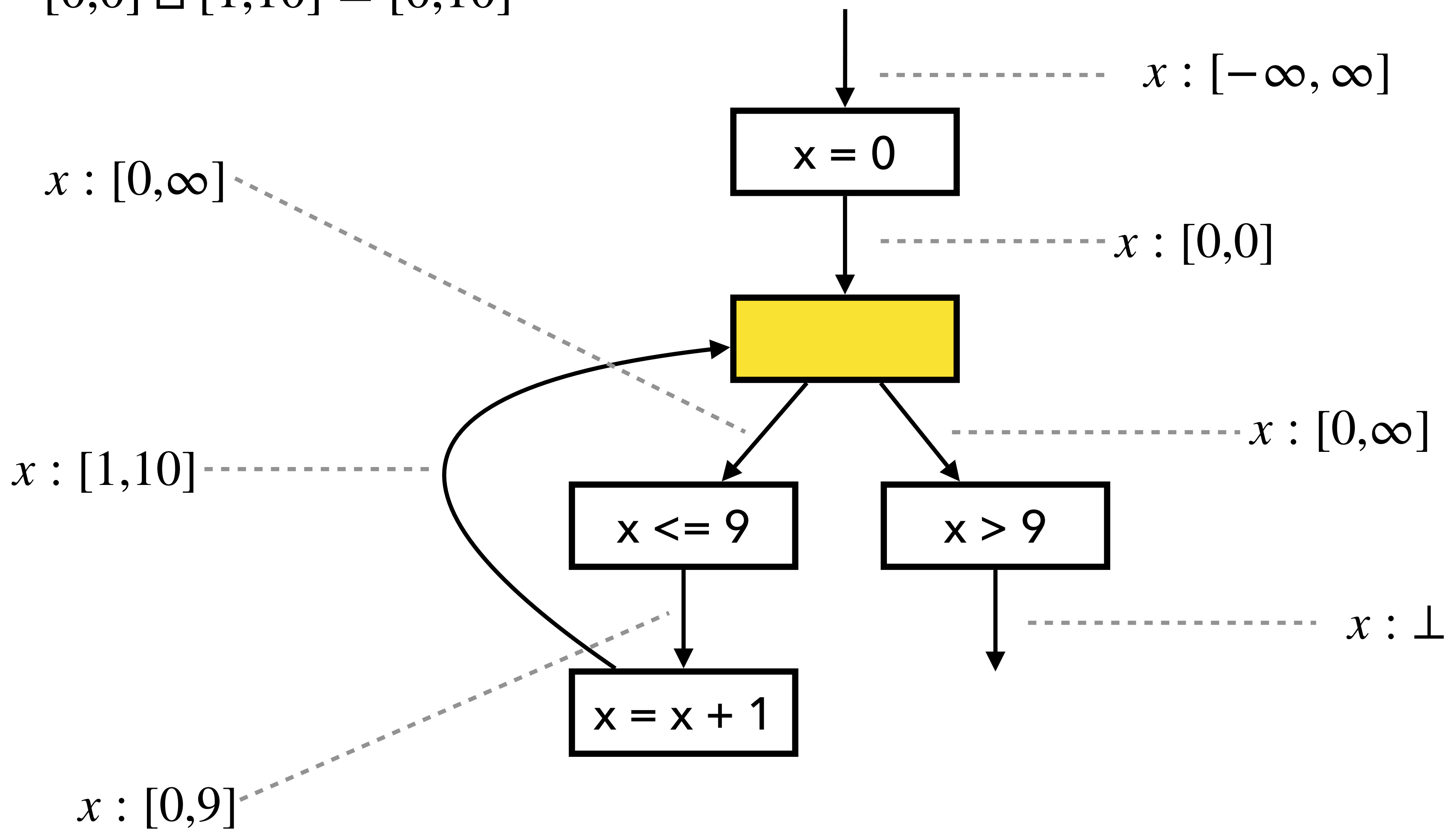
Fixed Point Comp. with Widening



Fixed Point Comp. with Widening

1. Compute output by joining inputs:

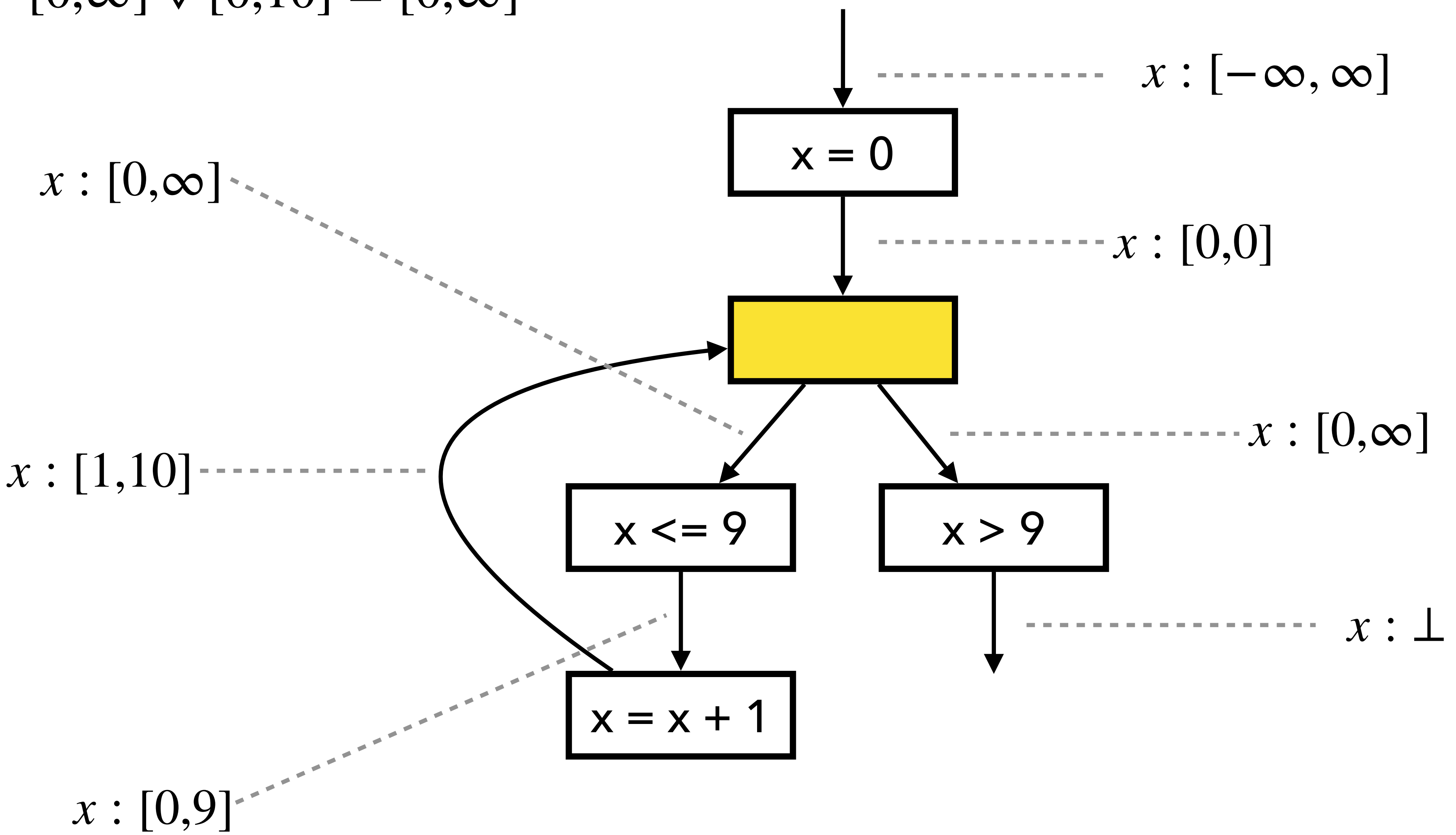
$$[0,0] \sqcup [1,10] = [0,10]$$



Fixed Point Comp. with Widening

2. Apply widening with old output:

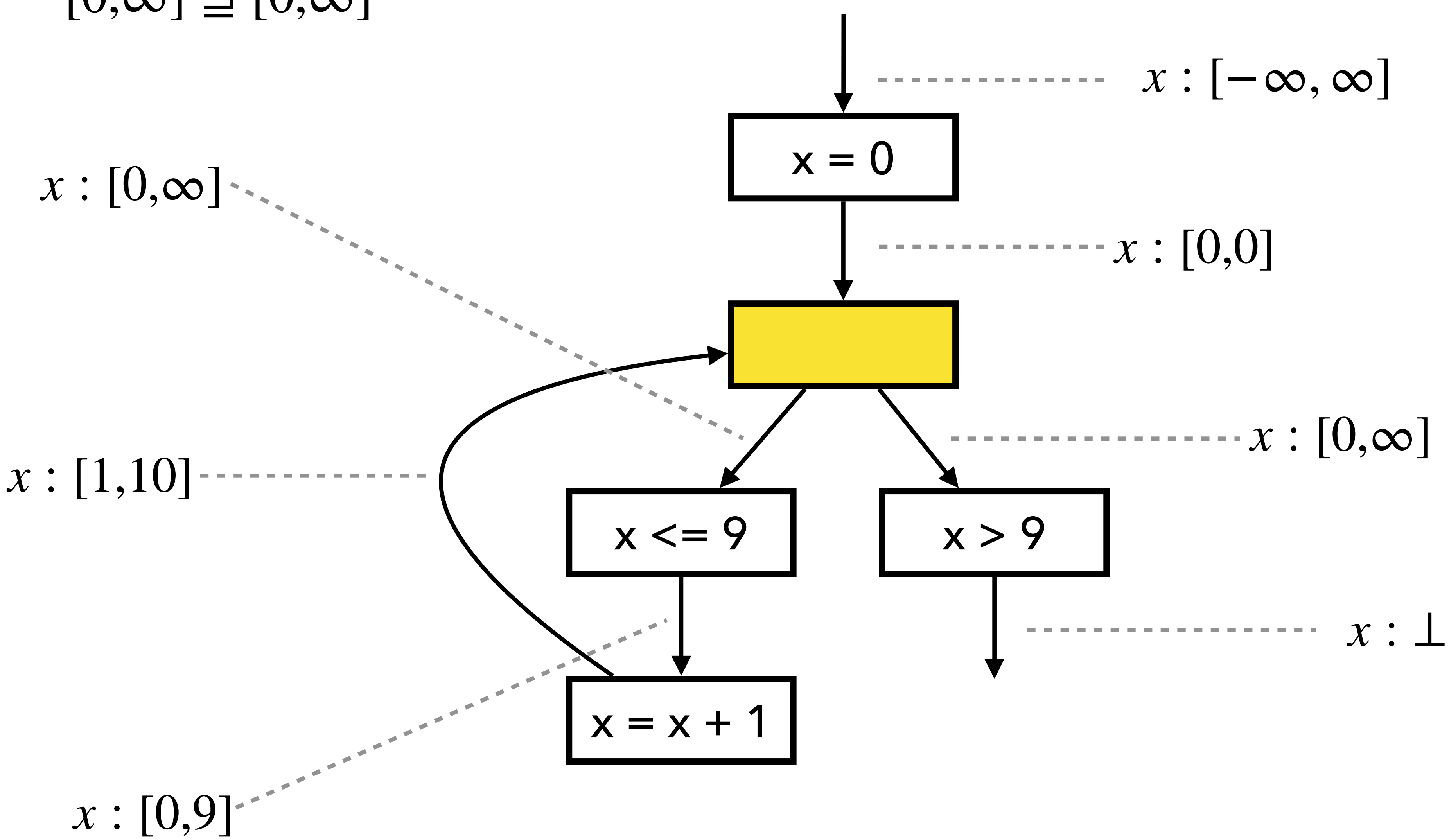
$$[0,\infty] \triangledown [0,10] = [0,\infty]$$



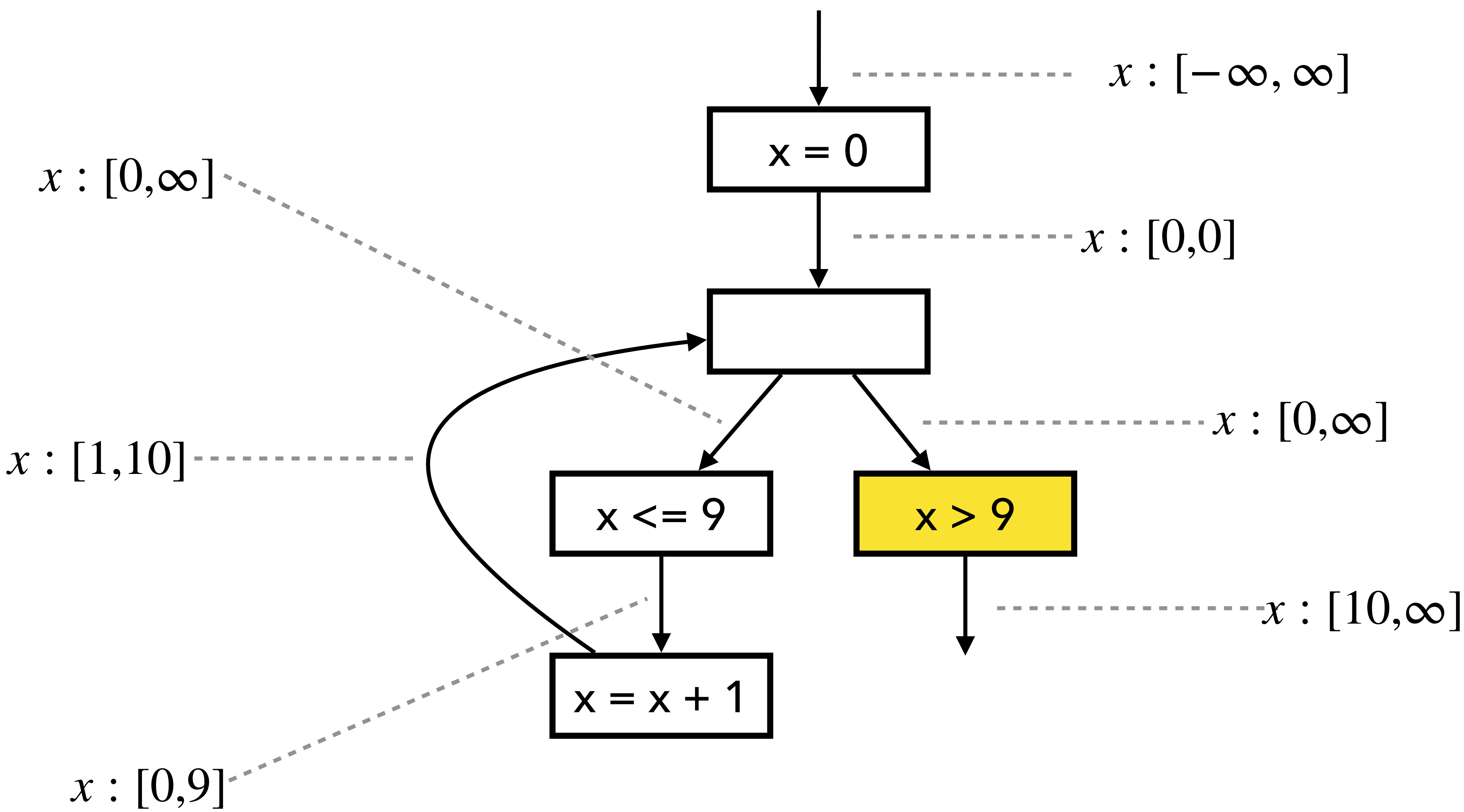
Fixed Point Comp. with Widening

3. Check if fixed point is reached

$$[0,\infty] \sqsupseteq [0,\infty]$$



Fixed Point Comp. with Widening

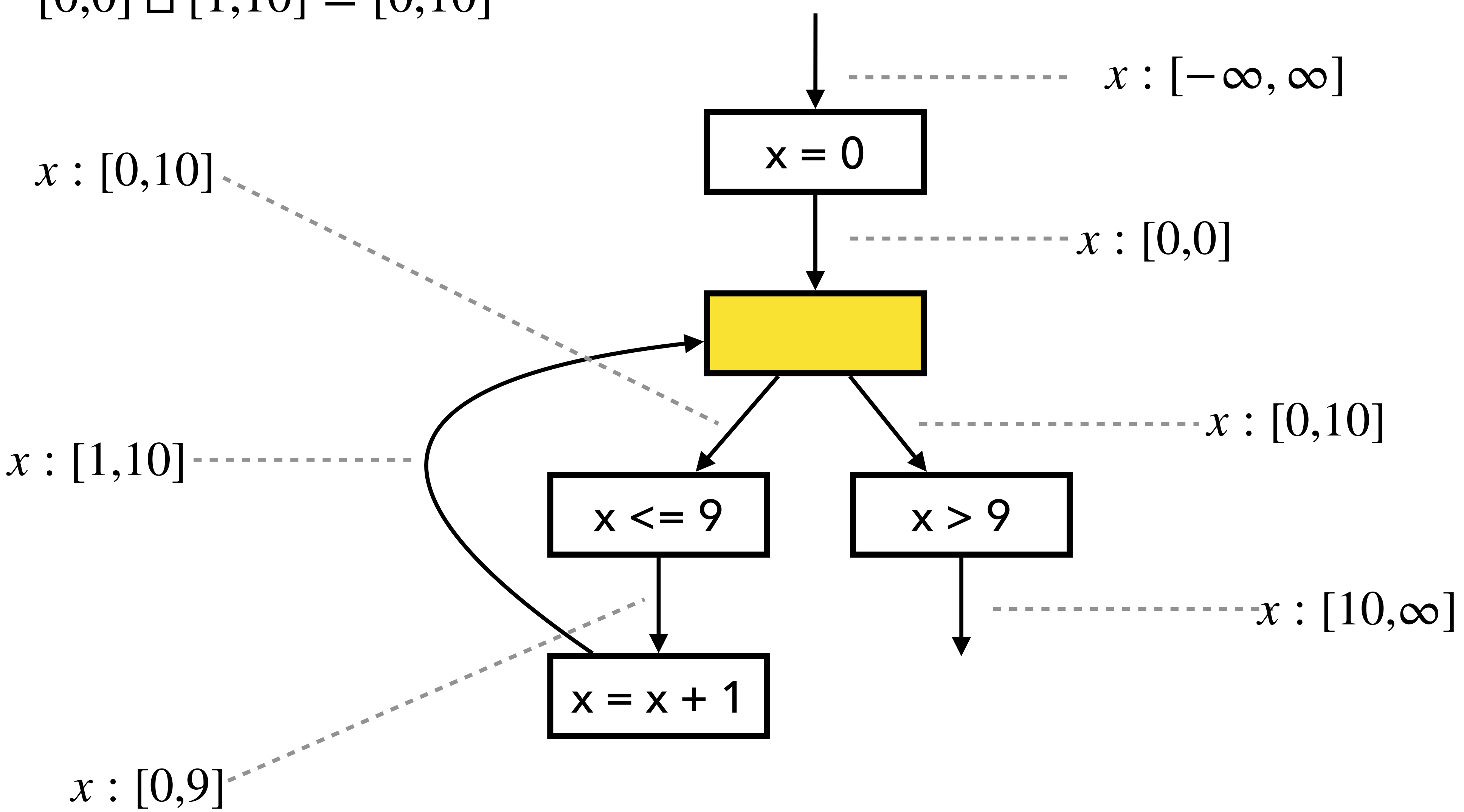


$$[0,\infty] \sqcap [10,\infty] = [10,\infty]$$

Fixed Point Comp. with Narrowing

1. Compute output by joining inputs:

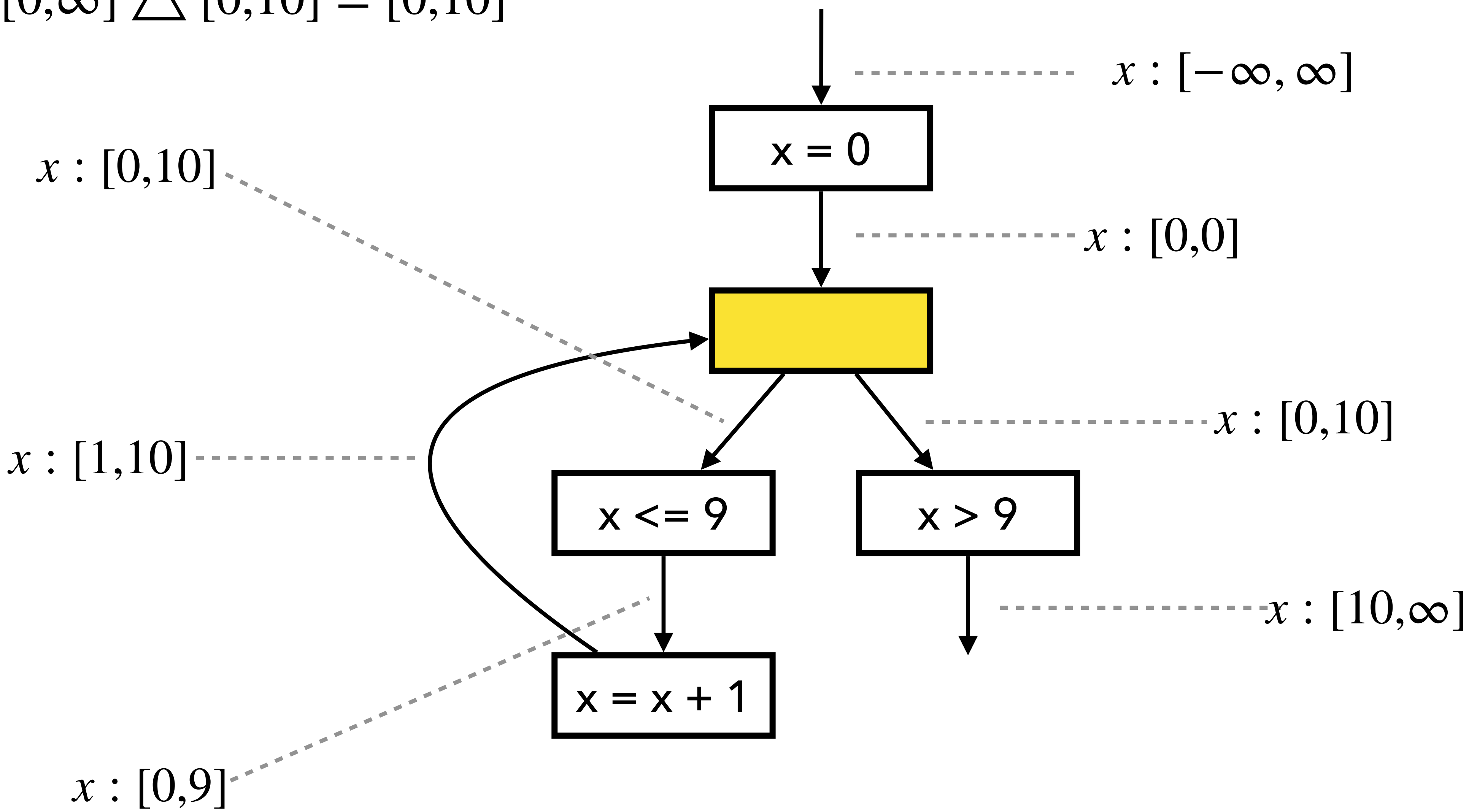
$$[0,0] \sqcup [1,10] = [0,10]$$



Fixed Point Comp. with Narrowing

2. Apply narrowing with old output:

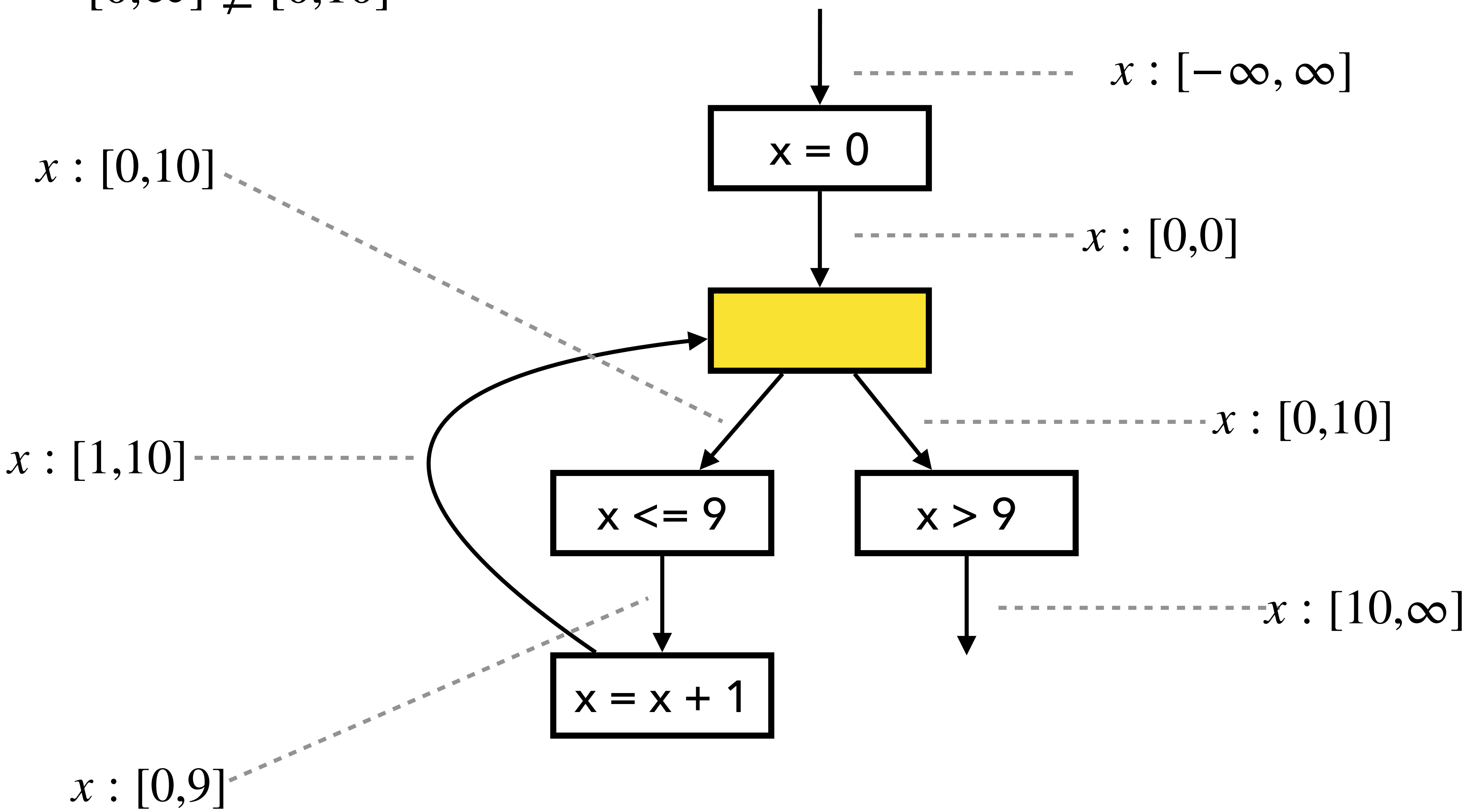
$$[0,\infty] \triangle [0,10] = [0,10]$$



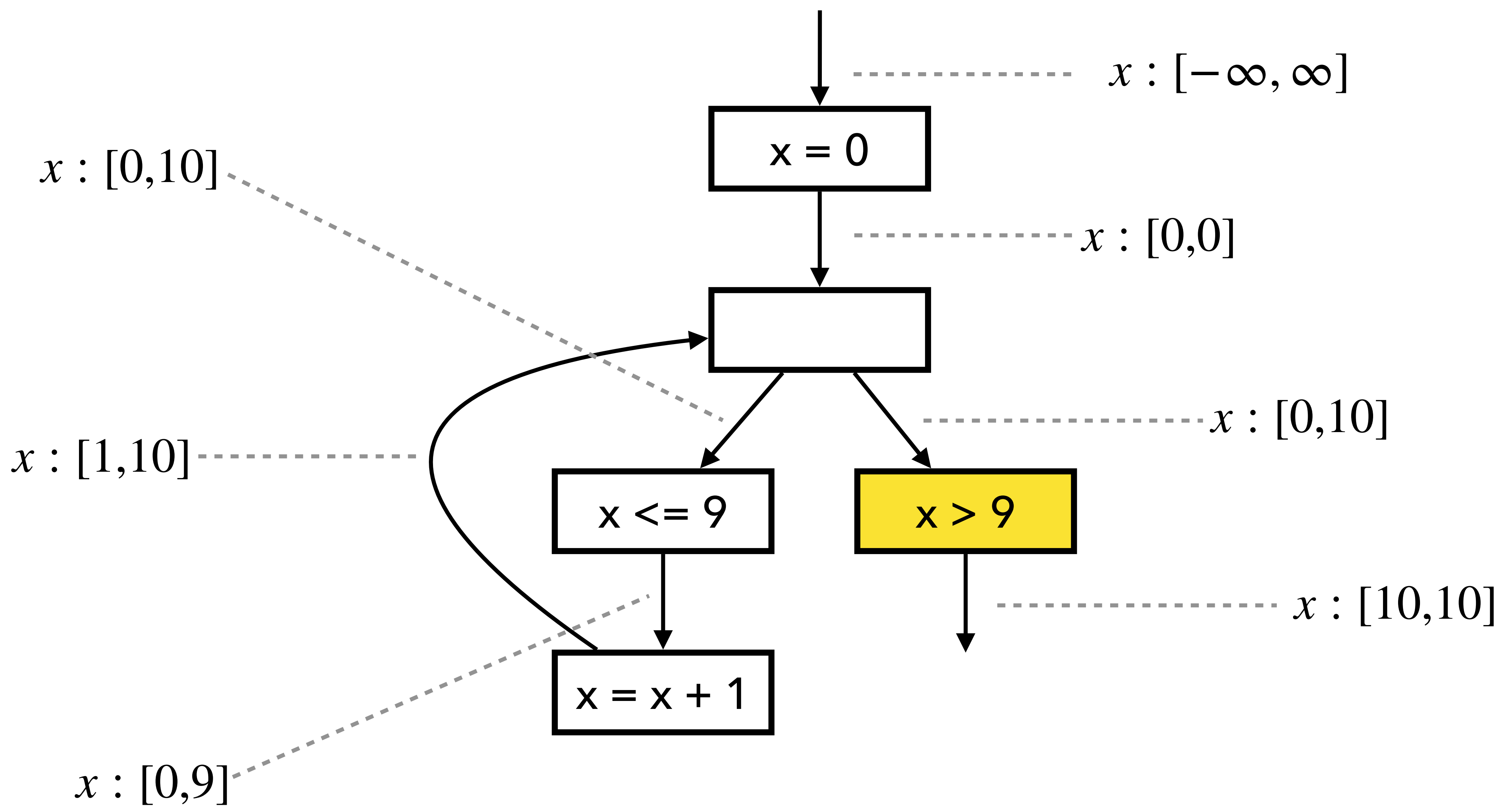
Fixed Point Comp. with Narrowing

3. Check if fixed point is reached:

$$[0,\infty] \not\sqsubseteq [0,10]$$



Fixed Point Comp. with Narrowing



The Interval Domain

- The set of intervals:

$$\hat{\mathbb{Z}} = \{\ \bot\ \} \cup \{[l, u] \mid l, u \in \mathbb{Z} \cup \{-\infty, \infty\}, l \leq u\}$$

- Partial order:

$$\bot \sqsubseteq \hat{z} \quad (\text{for any } \hat{z} \in \hat{\mathbb{Z}}) \qquad [l_1, u_1] \sqsubseteq [l_2, u_2] \iff l_2 \leq l_1 \wedge u_1 \leq u_2$$

- Join:

$$\bot \sqcup \hat{z} = \hat{z} \quad \hat{z} \sqcup \bot = \hat{z} \quad [l_1, u_1] \sqcup [l_2, u_2] = [\text{min}(l_1, l_2), \text{max}(u_1, u_2)]$$

- Meet:

$$[l_1, u_1] \sqcap [l_2, u_2] = [l_2, u_1] \quad (\text{if } l_1 \leq l_2 \wedge l_2 \leq u_1)$$

$$[l_1, u_1] \sqcap [l_2, u_2] = [l_1, u_2] \quad (\text{if } l_2 \leq l_1 \wedge l_1 \leq u_2)$$

$$\hat{z}_1 \sqcap \hat{z}_2 = \bot \quad (\text{otherwise})$$

The Interval Domain

- Widening:

$$\bot \bigtriangledown \hat{z} = \hat{z}$$
$$\hat{z} \bigtriangledown \bot = \hat{z}$$
$$[l_1, u_1] \bigtriangledown [l_2, u_2] = [l_1 > l_2 ? -\infty : l_1, u_1 < u_2 ? +\infty : u_1]$$

- Narrowing:

$$\bot \bigtriangleup \hat{z} = \bot$$
$$\hat{z} \bigtriangleup \bot = \bot$$
$$[l_1, u_1] \bigtriangleup [l_2, u_2] = [l_1 = -\infty ? l_2 : l_1, u_1 = +\infty ? u_2 : u_1]$$

The Interval Domain

- Addition / Subtraction / Multiplication:

$$[l_1, u_1] \hat{+} [l_2, u_2] = [l_1 + l_2, u_1 + u_2]$$

$$[l_1, u_1] \hat{-} [l_2, u_2] = [l_1 - u_2, u_1 - l_2]$$

$$[l_1, u_1] \hat{\times} [l_2, u_2] = [\text{min}(l_1l_2, l_1u_2, u_1l_2, u_1u_2), \text{max}(l_1l_2, l_1u_2, u_1l_2, u_1u_2)]$$

- Equality (=) produces ⊤ except for the cases:

$$[l_1, u_1] \hat{=} [l_2, u_2] = \hat{true} \quad (\text{if } l_1 = u_1 = l_2 = u_2)$$

$$[l_1, u_1] \hat{=} [l_2, u_2] = \hat{false} \quad (\text{no overlap})$$

- ``Less than" (<) produces ⊤ except for the cases:

$$[l_1, u_1] \hat{<} [l_2, u_2] = \hat{true} \quad (\text{if } u_1 < l_2)$$

$$[l_1, u_1] \hat{<} [l_2, u_2] = \hat{false} \quad (\text{if } l_1 > u_2)$$

Abstract Memory

$$\hat{\mathbb{M}} = \textbf{Var} \rightarrow \hat{\mathbb{Z}}$$

$$m_1 \sqsubseteq m_2 \iff \forall x \in \textbf{Var} \, . \, m_1(x) \sqsubseteq m_2(x)$$

$$m_1 \sqcup m_2 = \lambda x \, . \, m_1(x) \sqcup m_2(x)$$

$$m_1 \sqcap m_2 = \lambda x \, . \, m_1(x) \sqcap m_2(x)$$

$$m_1 \bigtriangledown m_2 = \lambda x \, . \, m_1(x) \bigtriangledown m_2(x)$$

$$m_1 \bigtriangleup m_2 = \lambda x \, . \, m_1(x) \bigtriangleup m_2(x)$$

Worklist Algorithm

Fixpoint comp. with widening

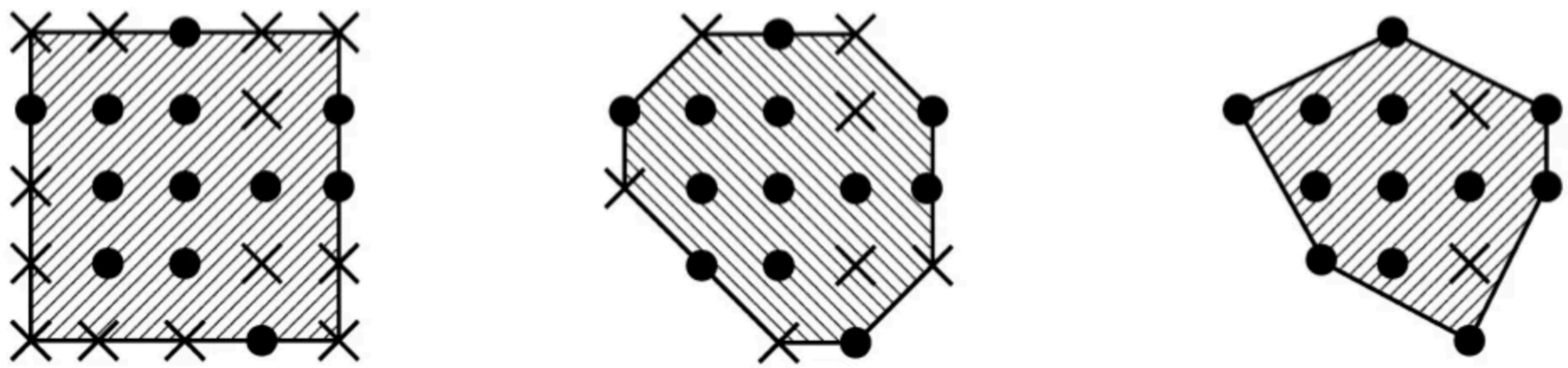
$W :=$ **Node**
$T := \lambda n\,.\,\bot_{\hat{\mathbb{M}}}$
$\textit{while}\ W \neq \emptyset$
$\quad n := \textit{choose}(W)$
$\quad W := W \setminus \{n\}$
$\quad \textit{in} := \textit{inputof}(n, T)$
$\quad \textit{out} := \textit{analyze}(n, \textit{in})$
$\quad \textit{if}\ \textit{out} \not\sqsubseteq T(n)$
$\quad\quad \textit{if widening is needed}$
$\quad\quad\quad T(n) := T(n) \bigtriangledown \textit{out}$
$\quad\quad \textit{else}$
$\quad\quad\quad T(n) := T(n) \sqcup \textit{out}$
$\quad\quad W := W \cup \textit{succ}(n)$

Fixpoint comp. with narrowing

$W :=$ **Node**
$\textit{while}\ W \neq \emptyset$
$\quad n := \textit{choose}(W)$
$\quad W := W \setminus \{n\}$
$\quad \textit{in} := \textit{inputof}(n, T)$
$\quad \textit{out} := \textit{analyze}(n, \textit{in})$
$\quad \textit{if}\ T(n) \not\sqsubseteq \textit{out}$
$\quad\quad T(n) := T(n) \bigtriangleup \textit{out}$
$\quad\quad W := W \cup \textit{succ}(n)$

Relational Abstract Domains

- Intervals vs. Octagons vs. Polyhedra



- Focus: Core idea of the Octagon domain*

```
int a[10];
x = 0; y = 0;

while (x < 9) {
    x++; y++;
}

a[y] = 0;
```

Octagon analysis

$x : [9,9]$
$y : [9,9]$
$x - y : [0,0]$
$x + y : [18,18]$

Interval analysis

$x : [9,9]$
$y : [0,\infty]$

*Antoine Miné. The Octagon Abstract Domain. https://arxiv.org/abs/cs/0703084

Difference Bound Matrix (DBM)

- $(N+1) \times (N+1)$ matrix (N: the number of variables): e.g.,

$$\begin{array}{c} \\ 0 \\ x \\ y \end{array} \begin{array}{c} \begin{array}{ccc} 0 & x & y \end{array} \\ \begin{bmatrix} 0-0 & x-0 & y-0 \\ 0-x & x-x & y-x \\ 0-y & x-y & y-y \end{bmatrix} \end{array}$$

- Example

$$\begin{bmatrix} 0 & 10 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Longleftrightarrow \begin{array}{c} 0 \le x \le 10 \\ 0 \le y \le 10 \\ y - x \le 0 \\ x - y \le 0 \end{array} \qquad \begin{bmatrix} 0 & 10 & +\infty \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \Longleftrightarrow \begin{array}{c} 1 \le x \le 10 \\ 0 \le y \\ y - x \le -1 \\ x - y \le 1 \end{array}$$

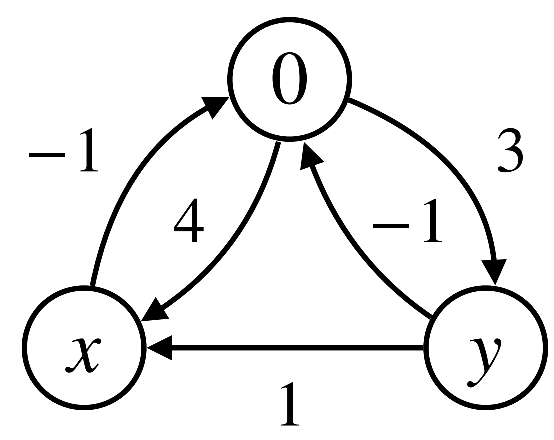
Difference Bound Matrix (DBM)

- A DBM represents a set of program states (N-dim points)

$$\gamma\left(\begin{bmatrix} 0 & 10 & +\infty \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\right) = \{(x, y) \mid 1 \leq x \leq 10, 0 \leq y, y - x \leq -1, x - y \leq 1\}$$

- A DBM can also be represented by a directed graph

$$\begin{array}{c} \\ 0 \\ x \\ y \end{array} \begin{array}{c} \begin{array}{ccc} 0 & x & y \end{array} \\ \begin{bmatrix} +\infty & 4 & 3 \\ -1 & +\infty & +\infty \\ -1 & 1 & +\infty \end{bmatrix} \end{array} \iff$$



Difference Bound Matrix (DBM)

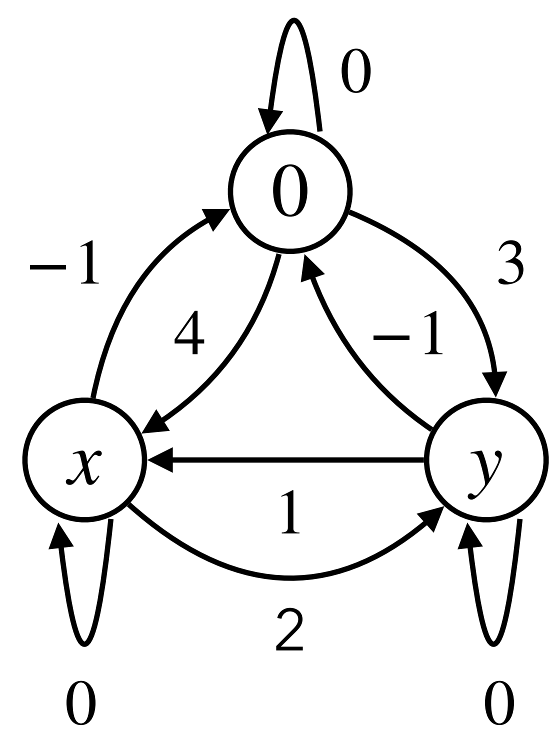
- Two different DBMs can represent the same set of points

$$\gamma\left(\begin{bmatrix} +\infty & 4 & 3 \\ -1 & +\infty & +\infty \\ -1 & 1 & +\infty \end{bmatrix}\right) = \gamma\left(\begin{bmatrix} 0 & 5 & 3 \\ -1 & +\infty & +\infty \\ -1 & 1 & +\infty \end{bmatrix}\right)$$

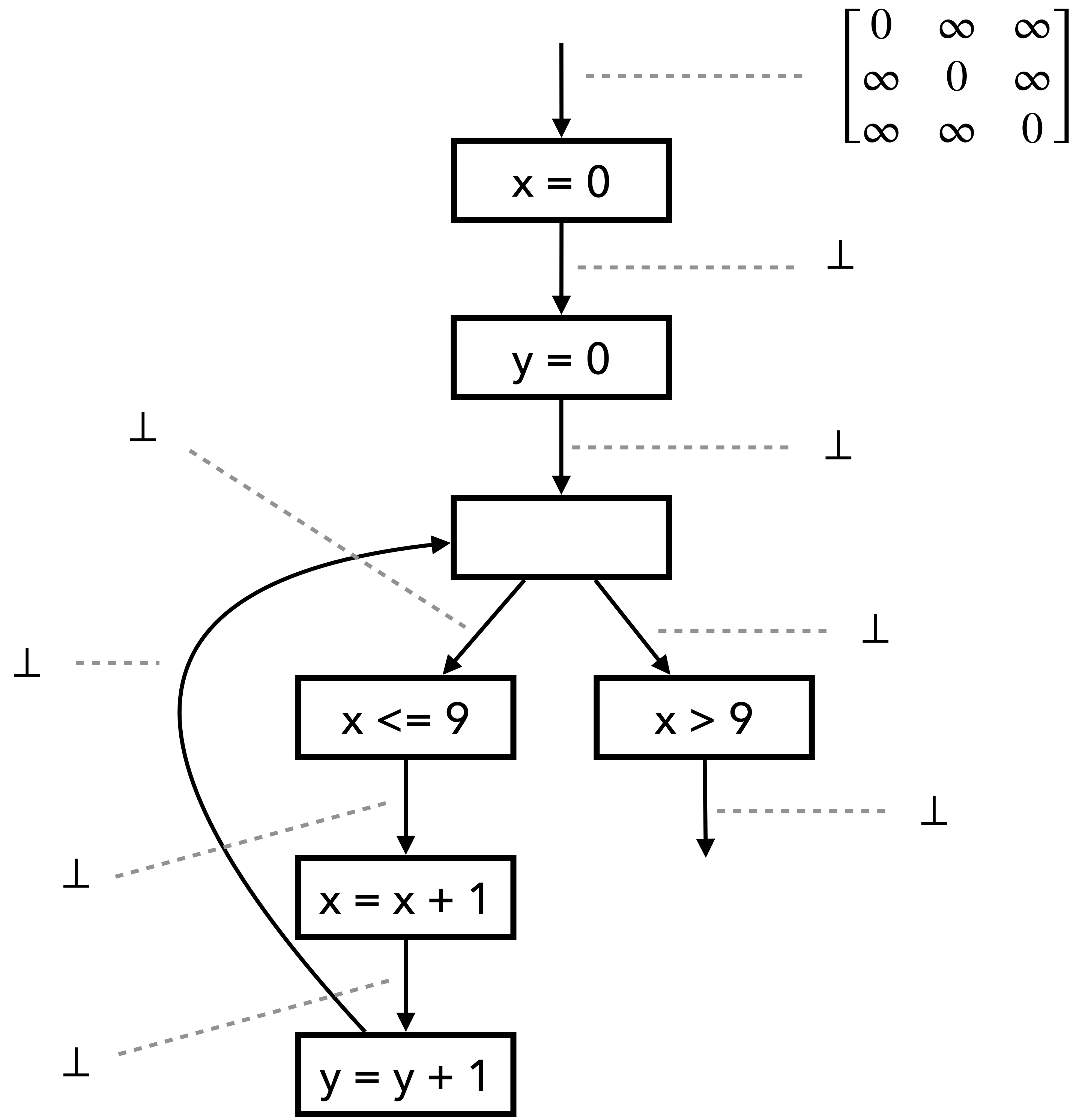
- Closure (normalization) via the Floyd-Warshall algorithm

$$\begin{bmatrix} +\infty & 4 & 3 \\ -1 & +\infty & +\infty \\ -1 & 1 & +\infty \end{bmatrix}^* = \begin{bmatrix} 0 & 4 & 3 \\ -1 & 0 & 2 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 5 & 3 \\ -1 & +\infty & +\infty \\ -1 & 1 & +\infty \end{bmatrix}^* = \begin{bmatrix} 0 & 4 & 3 \\ -1 & 0 & 2 \\ -1 & 1 & 0 \end{bmatrix}$$



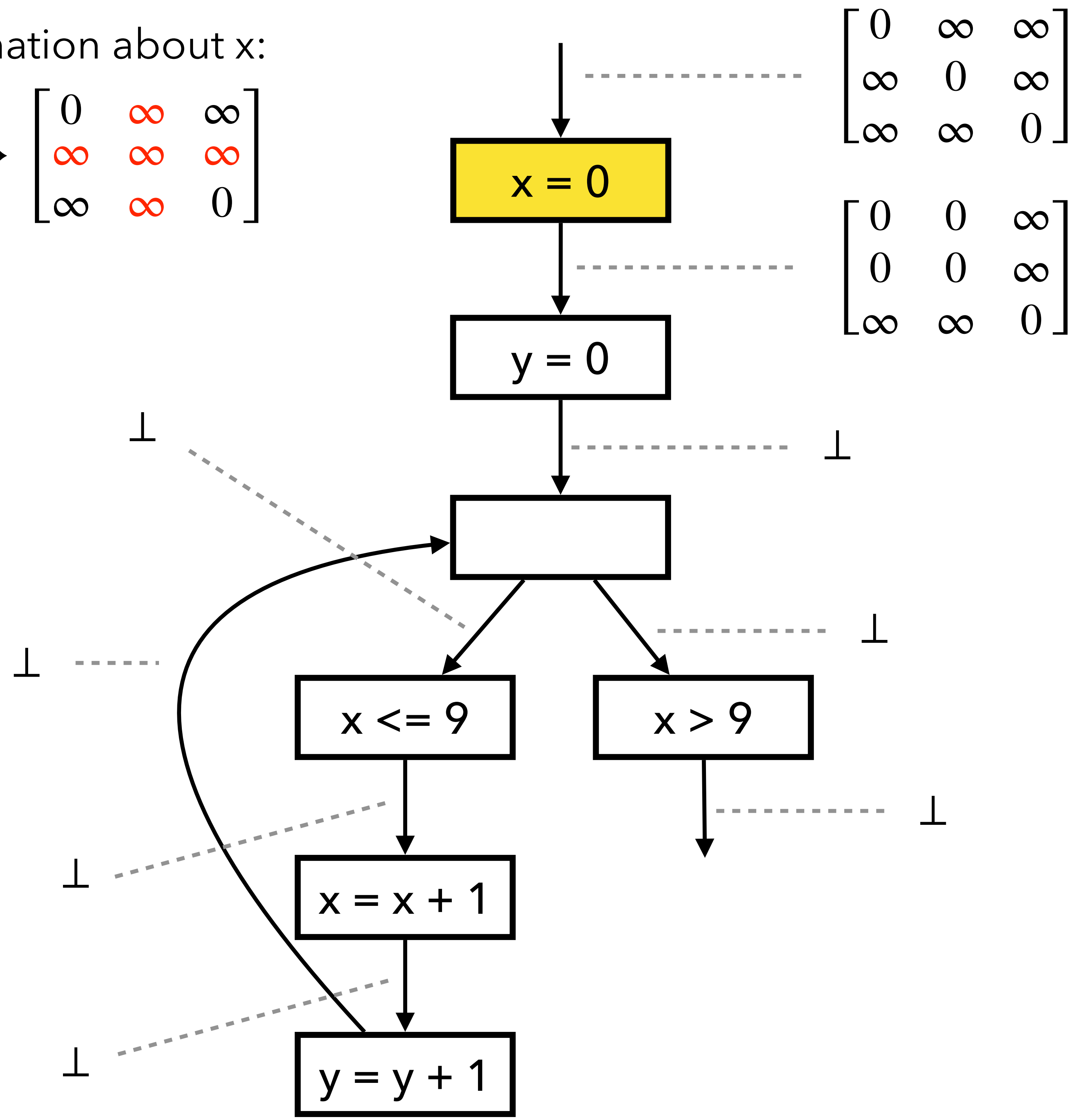
Fixed Point Comp. with Widening



Fixed Point Comp. with Widening

1. Remove information about x:

$$\begin{bmatrix} 0 & \infty & \infty \\ \infty & 0 & \infty \\ \infty & \infty & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & \infty & \infty \\ \infty & \infty & \infty \\ \infty & \infty & 0 \end{bmatrix}$$

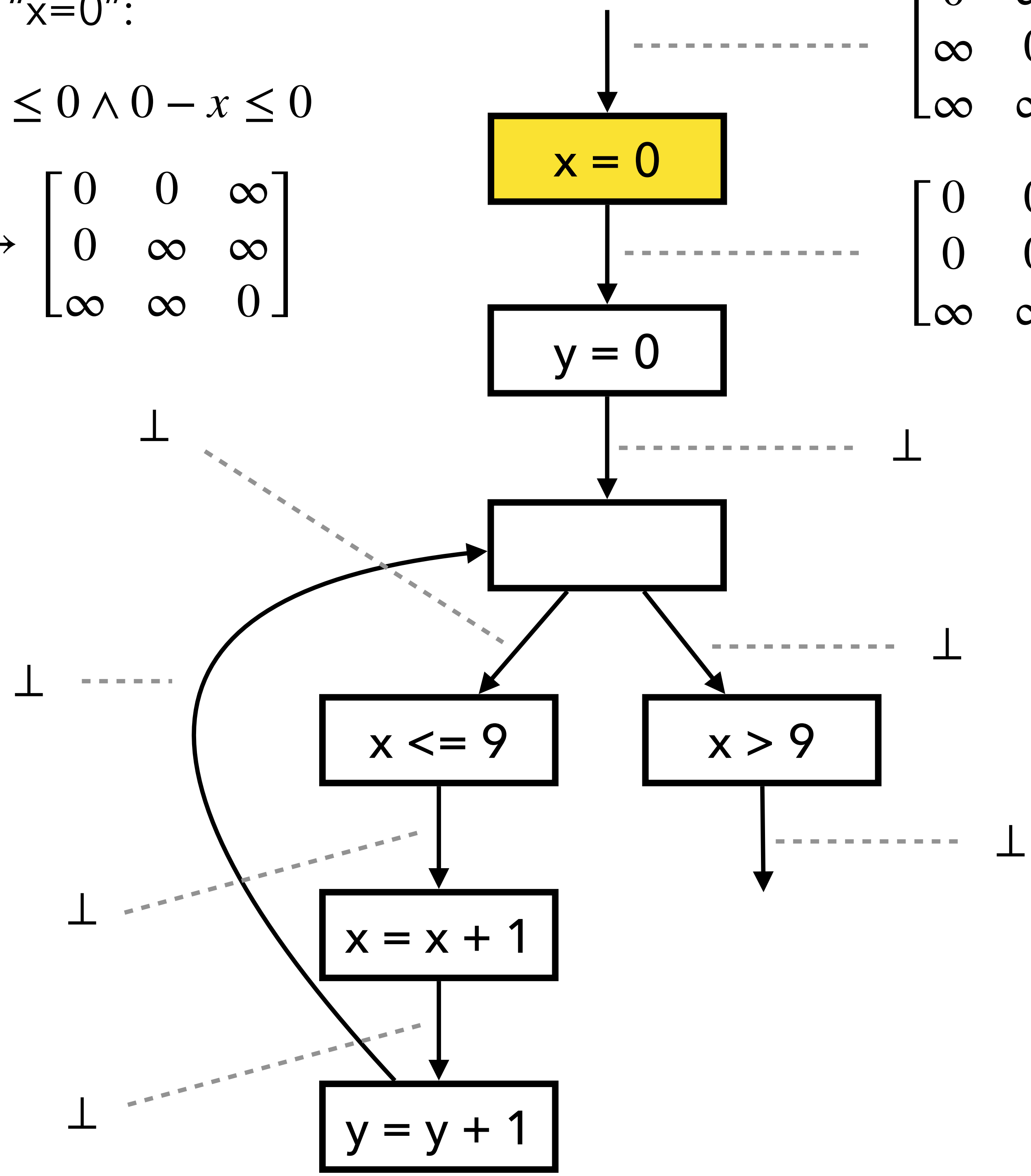


Fixed Point Comp. with Widening

2. Add constraint “x=0”:

$$x = 0 \iff x - 0 \le 0 \land 0 - x \le 0$$

$$\begin{bmatrix} 0 & \infty & \infty \\ \infty & \infty & \infty \\ \infty & \infty & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & \infty \\ 0 & \infty & \infty \\ \infty & \infty & 0 \end{bmatrix}$$



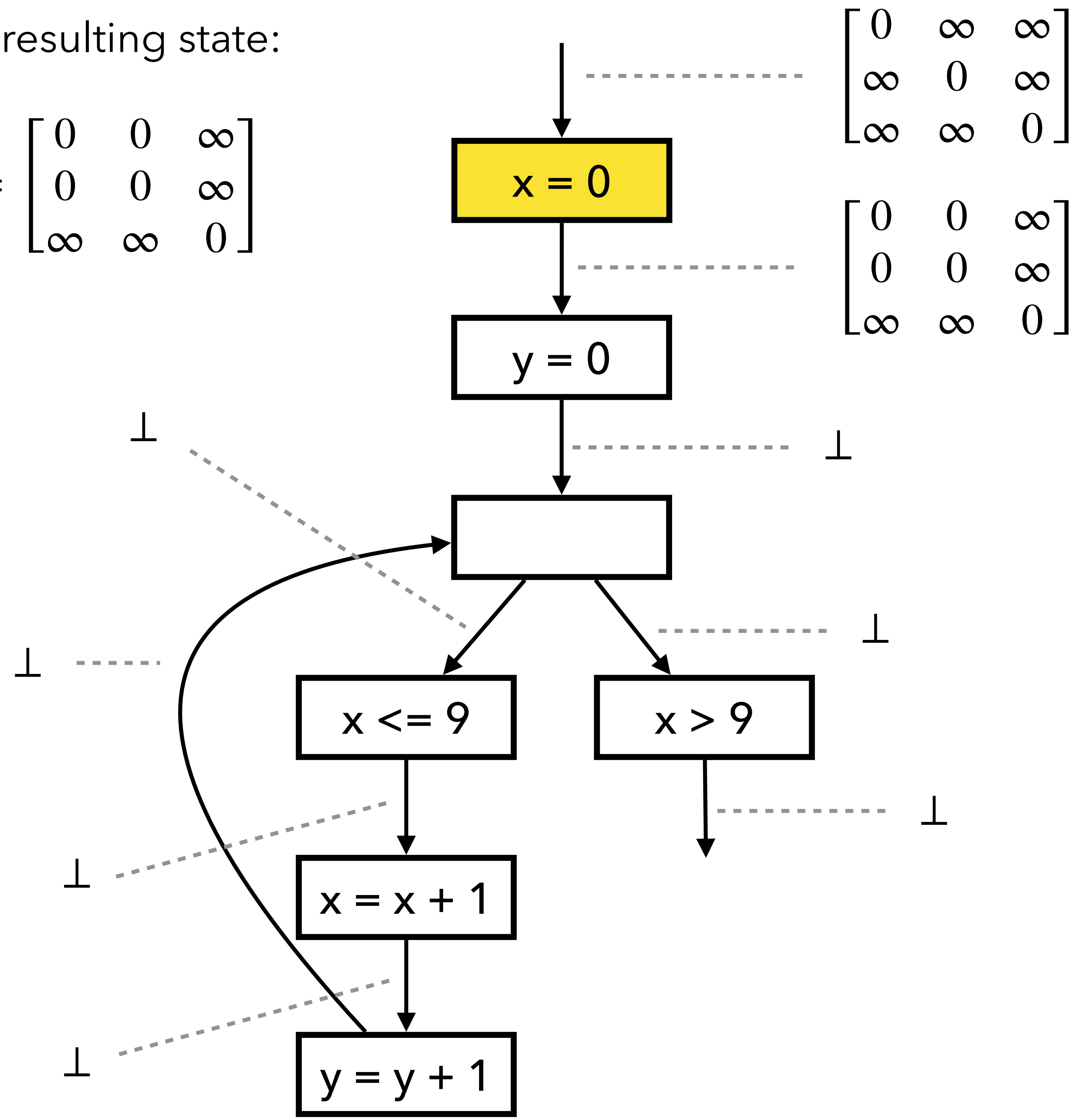
$$\begin{bmatrix} 0 & \infty & \infty \\ \infty & 0 & \infty \\ \infty & \infty & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & \infty \\ 0 & 0 & \infty \\ \infty & \infty & 0 \end{bmatrix}$$

Fixed Point Comp. with Widening

3. Normalize the resulting state:

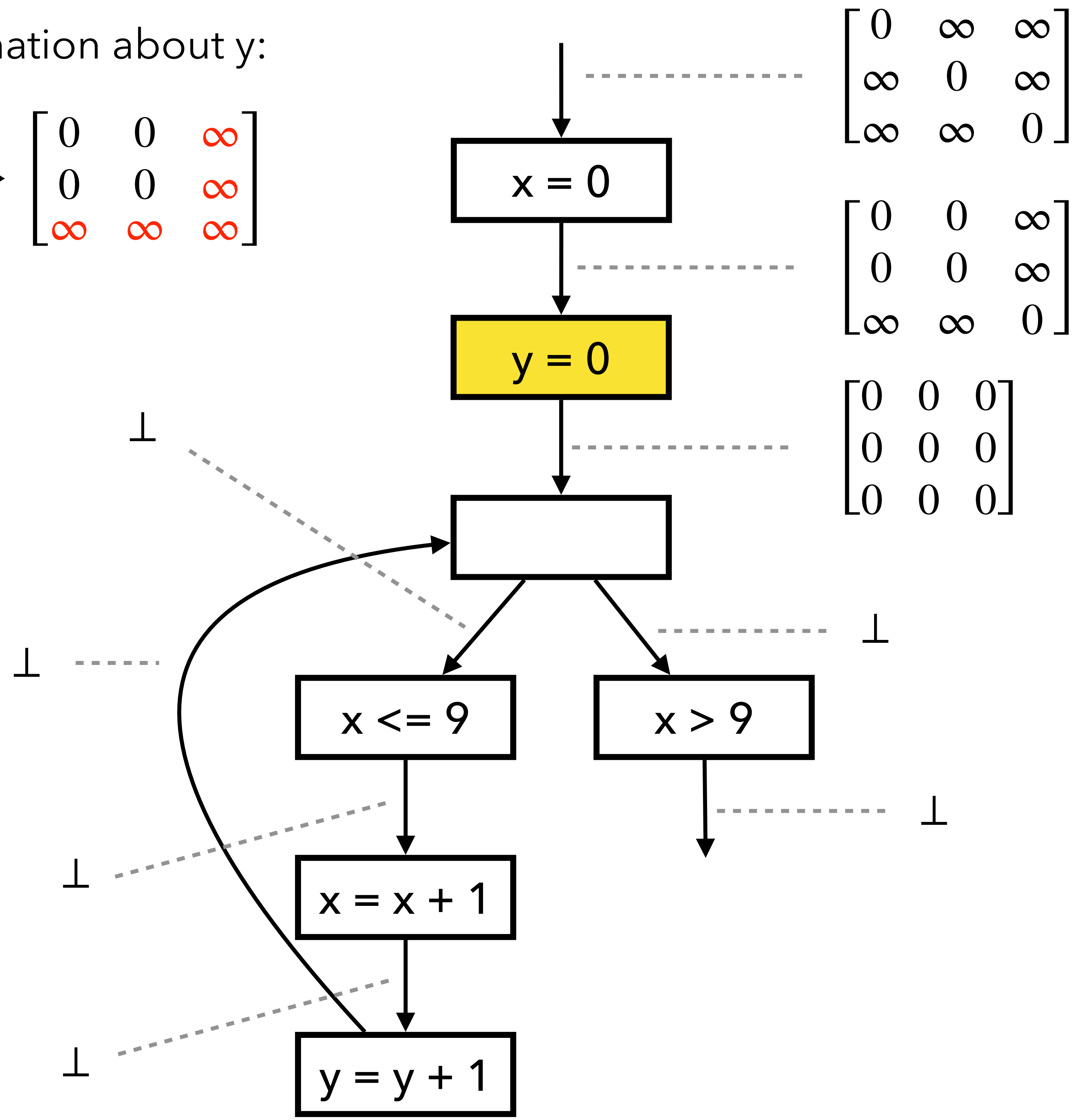
$$\begin{bmatrix} 0 & 0 & \infty \\ 0 & \infty & \infty \\ \infty & \infty & 0 \end{bmatrix}^* = \begin{bmatrix} 0 & 0 & \infty \\ 0 & 0 & \infty \\ \infty & \infty & 0 \end{bmatrix}$$



Fixed Point Comp. with Widening

1. Remove information about y:

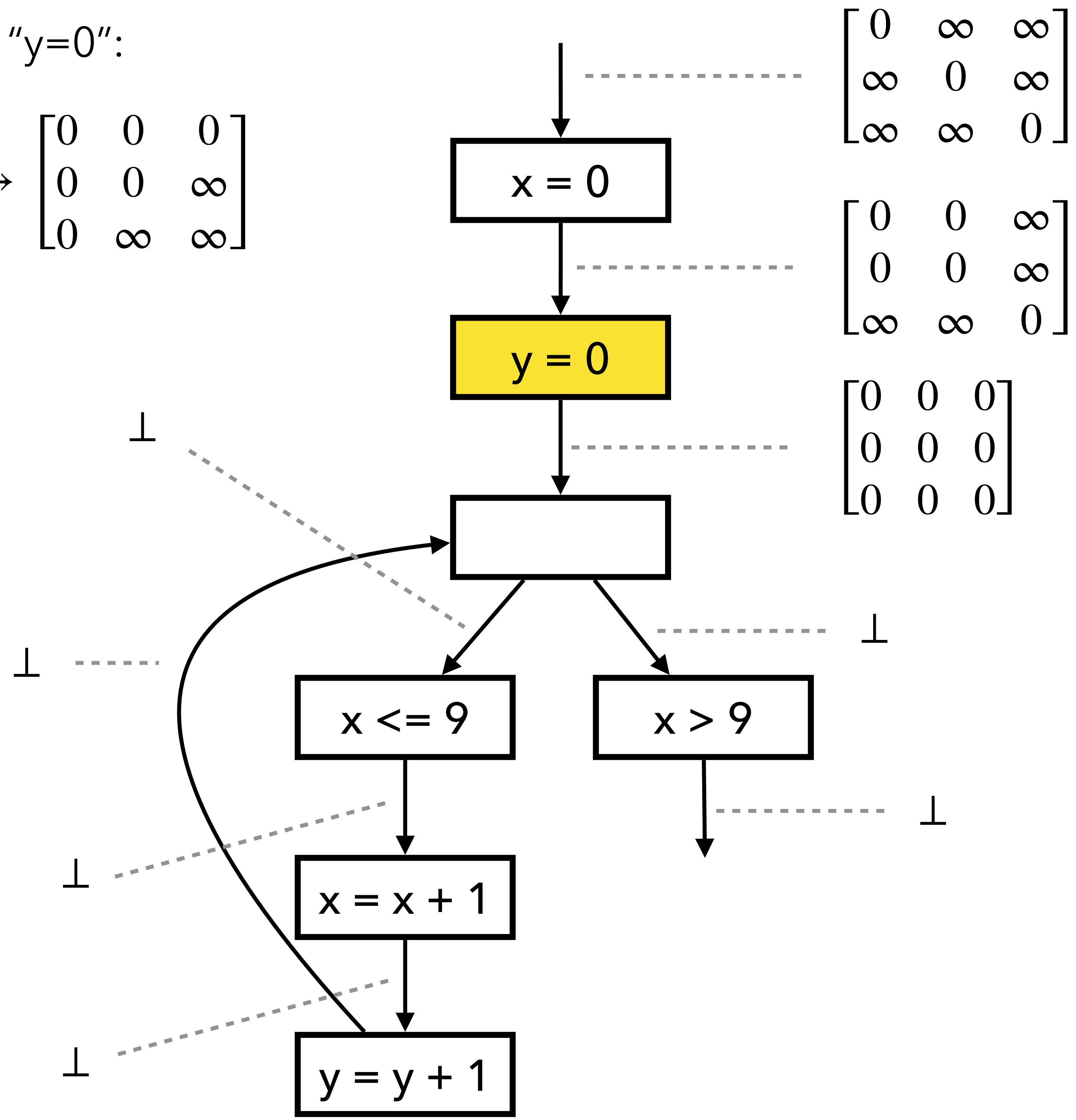
$$\begin{bmatrix}0 & 0 & \infty\\ 0 & 0 & \infty\\ \infty & \infty & 0\end{bmatrix} \rightarrow \begin{bmatrix}0 & 0 & \infty\\ 0 & 0 & \infty\\ \infty & \infty & \infty\end{bmatrix}$$



Fixed Point Comp. with Widening

2. Add constraint "y=0":

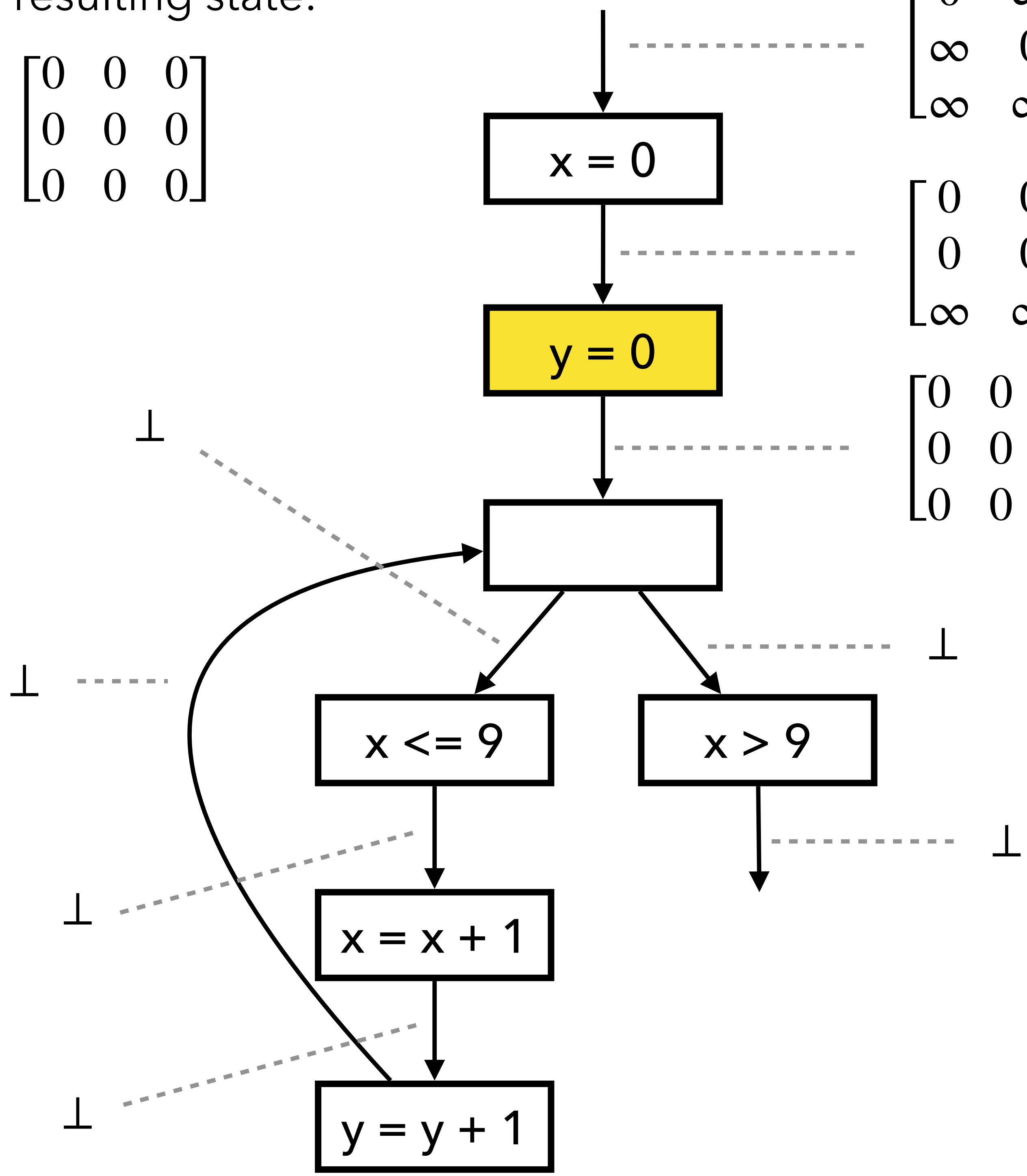
$$\begin{bmatrix} 0 & 0 & \infty \\ 0 & 0 & \infty \\ \infty & \infty & \infty \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \infty \\ 0 & \infty & \infty \end{bmatrix}$$



Fixed Point Comp. with Widening

3. Normalize the resulting state:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \infty \\ 0 & \infty & \infty \end{bmatrix}^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

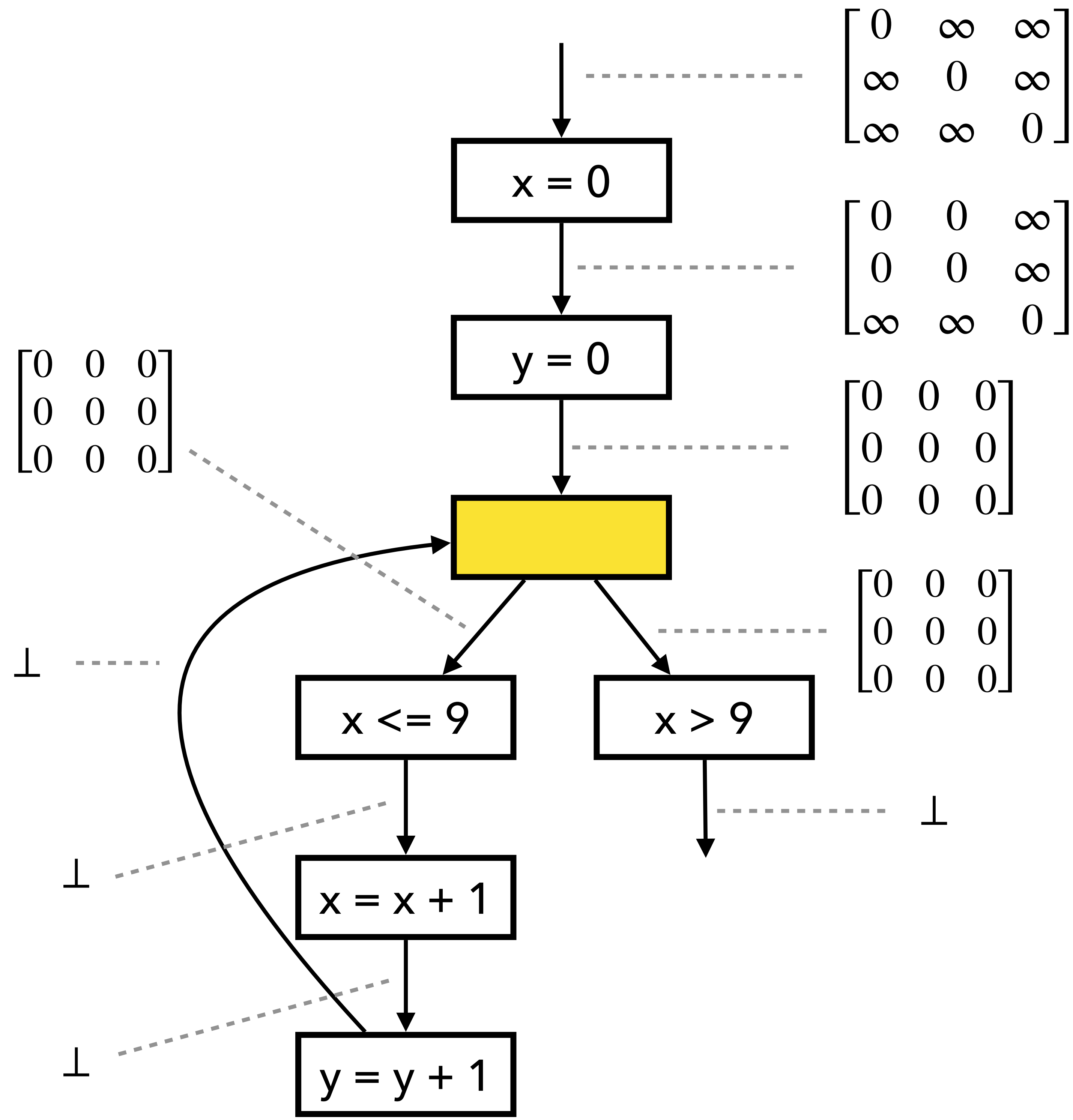


$$\begin{bmatrix} 0 & \infty & \infty \\ \infty & 0 & \infty \\ \infty & \infty & 0 \end{bmatrix}$$

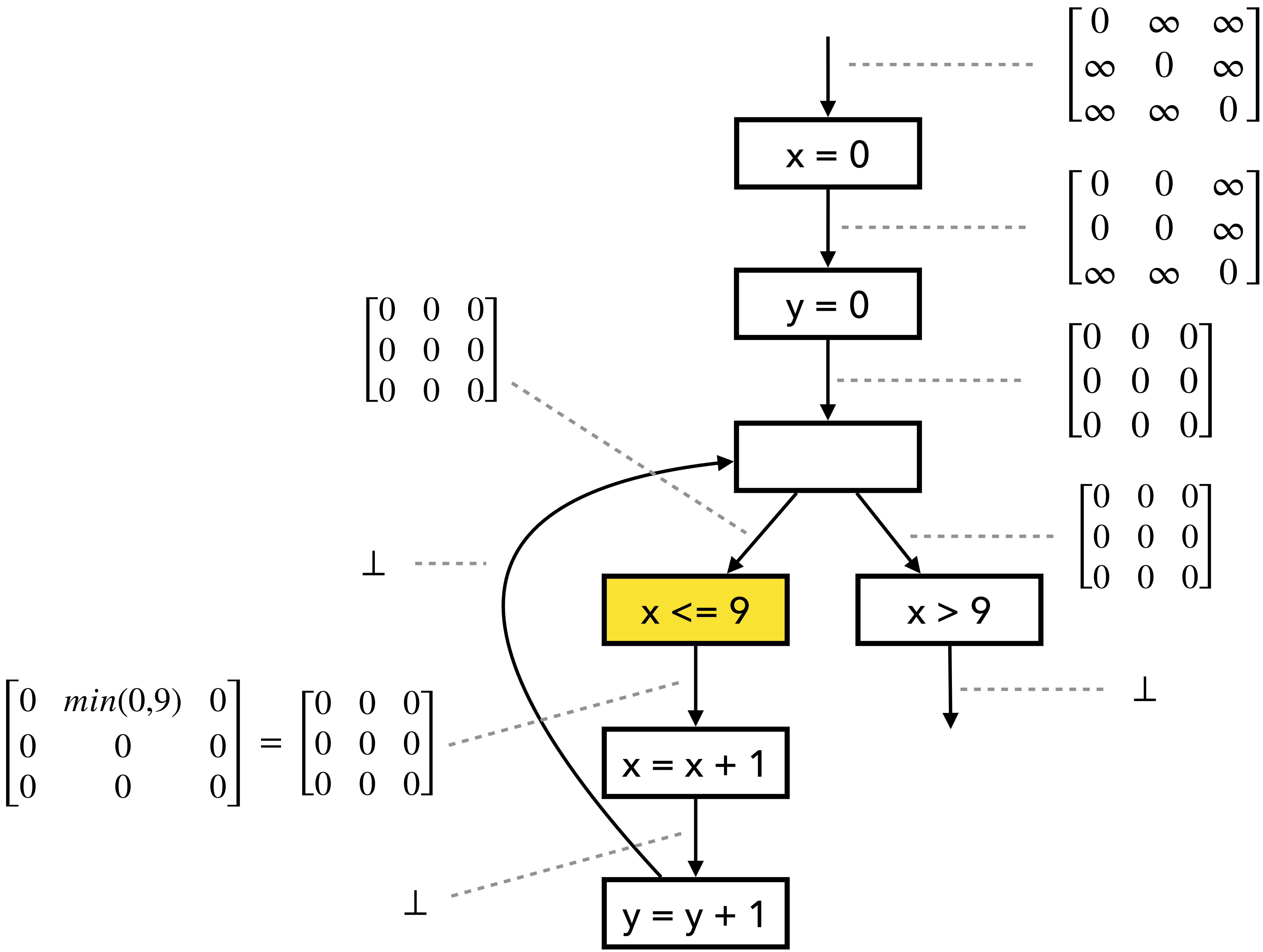
$$\begin{bmatrix} 0 & 0 & \infty \\ 0 & 0 & \infty \\ \infty & \infty & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

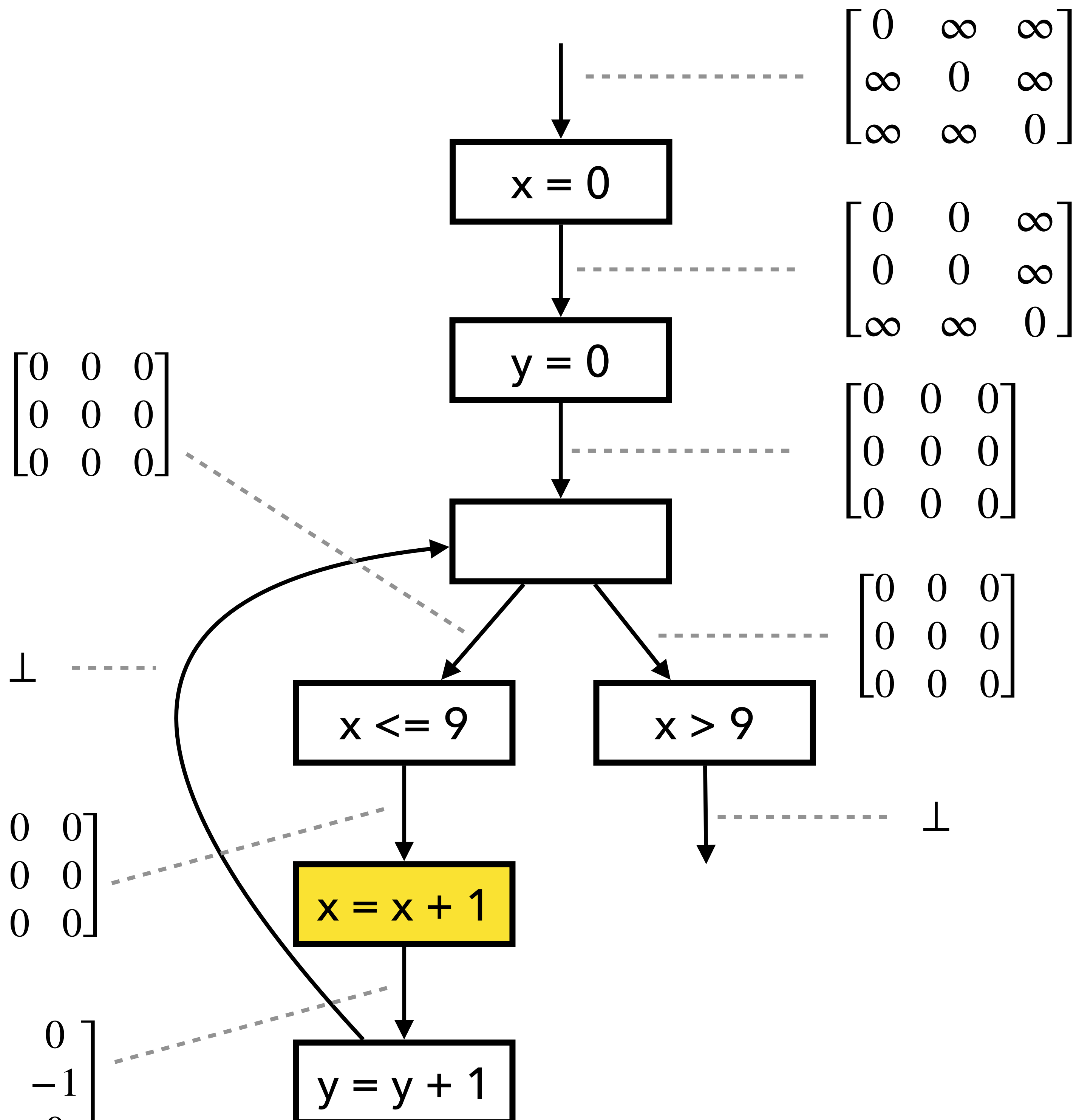
Fixed Point Comp. with Widening



Fixed Point Comp. with Widening



Fixed Point Comp. with Widening



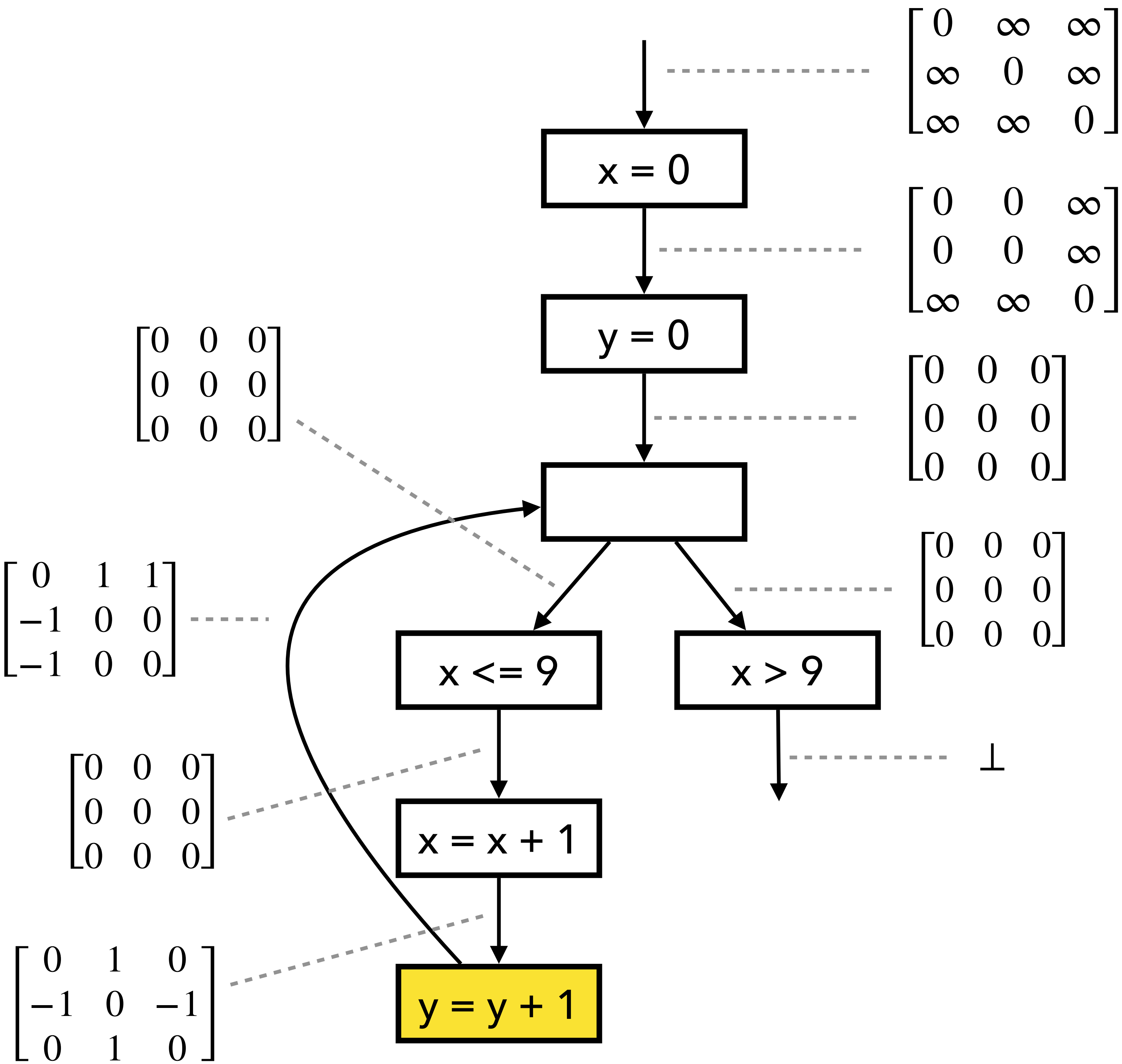
$$\begin{bmatrix}0&\infty&\infty\\\infty&0&\infty\\\infty&\infty&0\end{bmatrix}\quad \begin{bmatrix}0&0&\infty\\0&0&\infty\\\infty&\infty&0\end{bmatrix}\quad \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix} \to \begin{bmatrix}0&1&0\\-1&0&-1\\0&1&0\end{bmatrix}$$

$$x - x' \leq c \to x - x' \leq c + 1$$

$$x' - x \leq c \to x' - x \leq c - 1$$

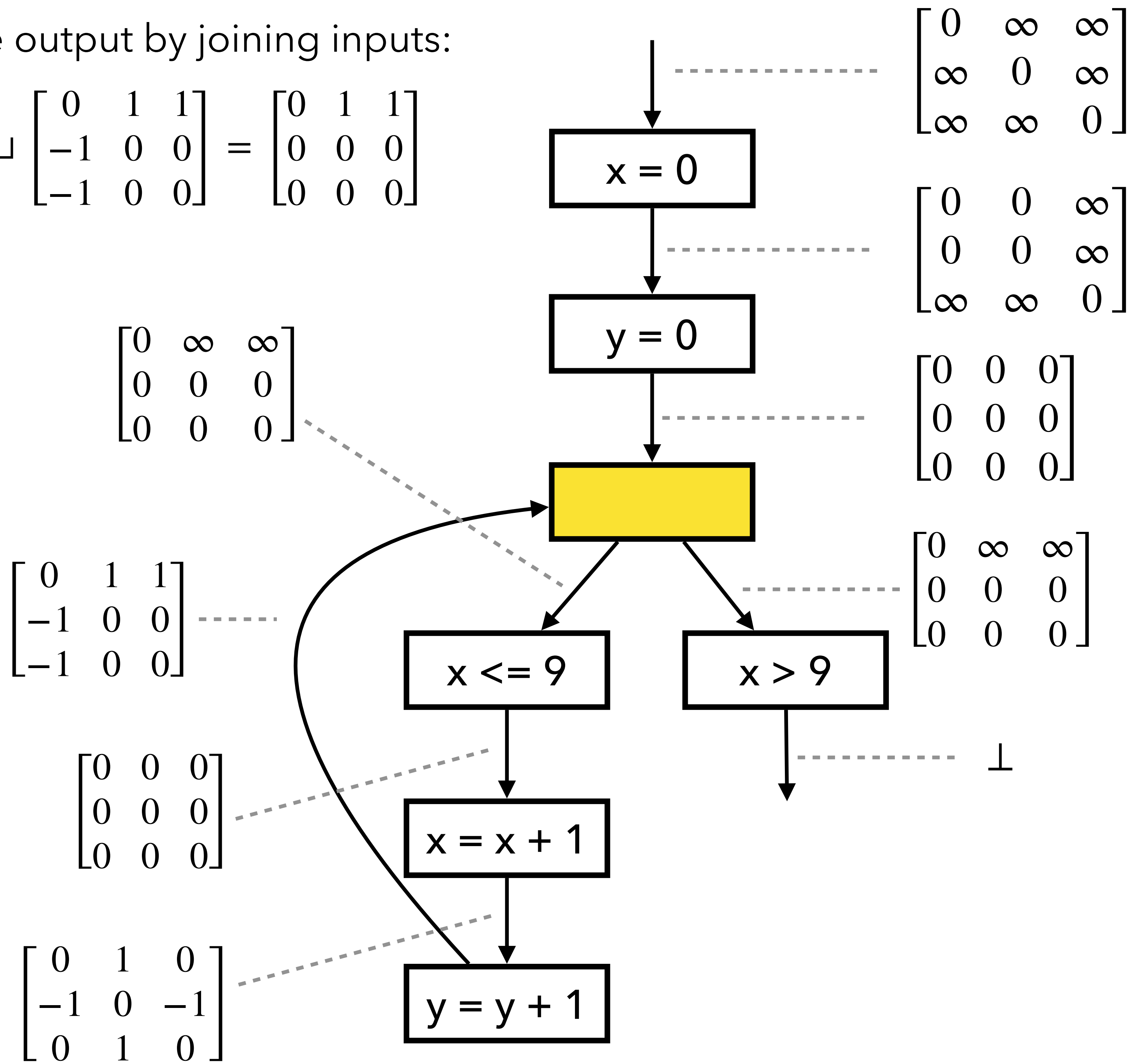
Fixed Point Comp. with Widening



Fixed Point Comp. with Widening

1. Compute output by joining inputs:

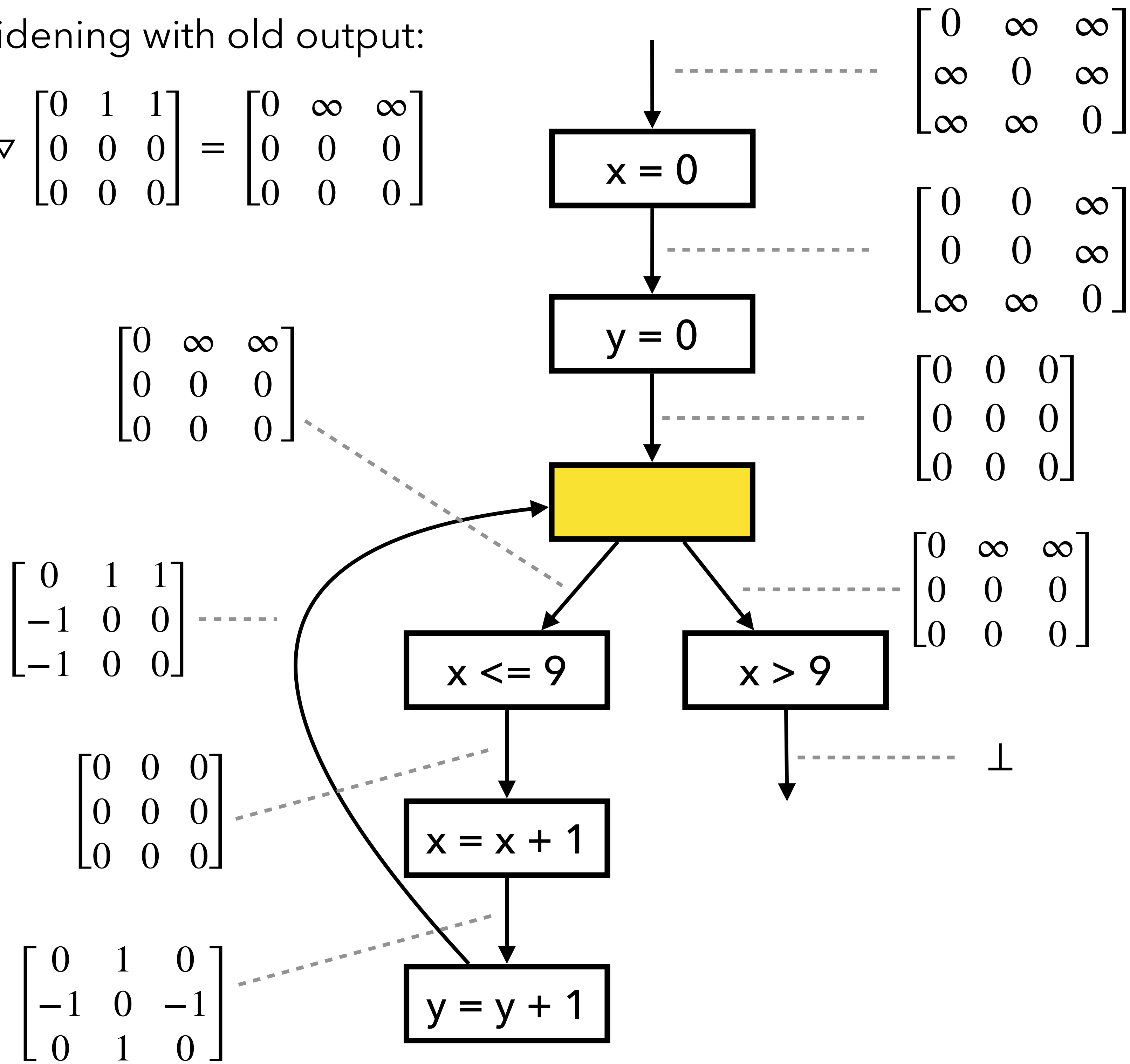
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sqcup \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Fixed Point Comp. with Widening

2. Apply widening with old output:

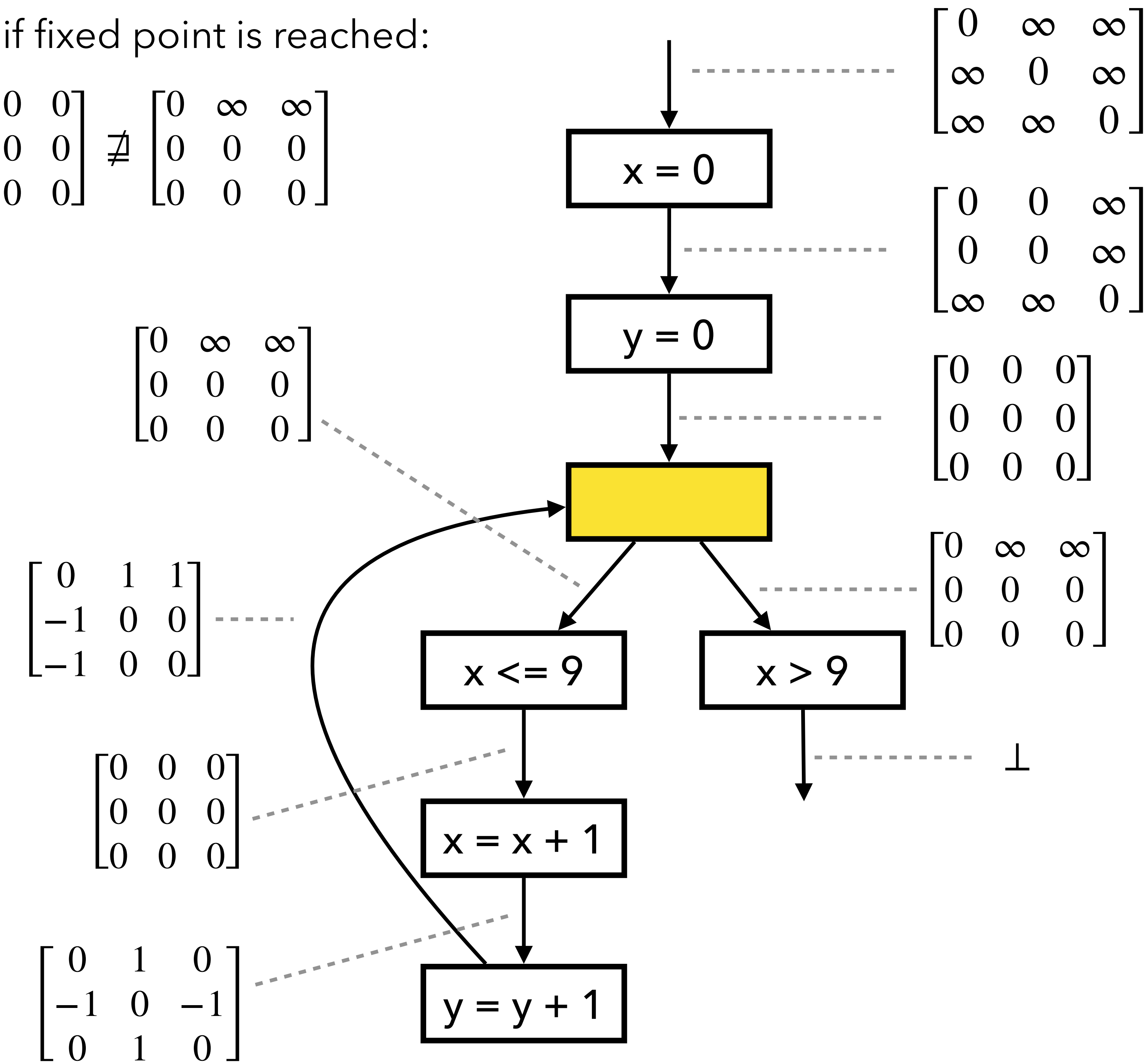
$$\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} \triangledown \begin{bmatrix}0 & 1 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & \infty & \infty\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$



Fixed Point Comp. with Widening

3. Check if fixed point is reached:

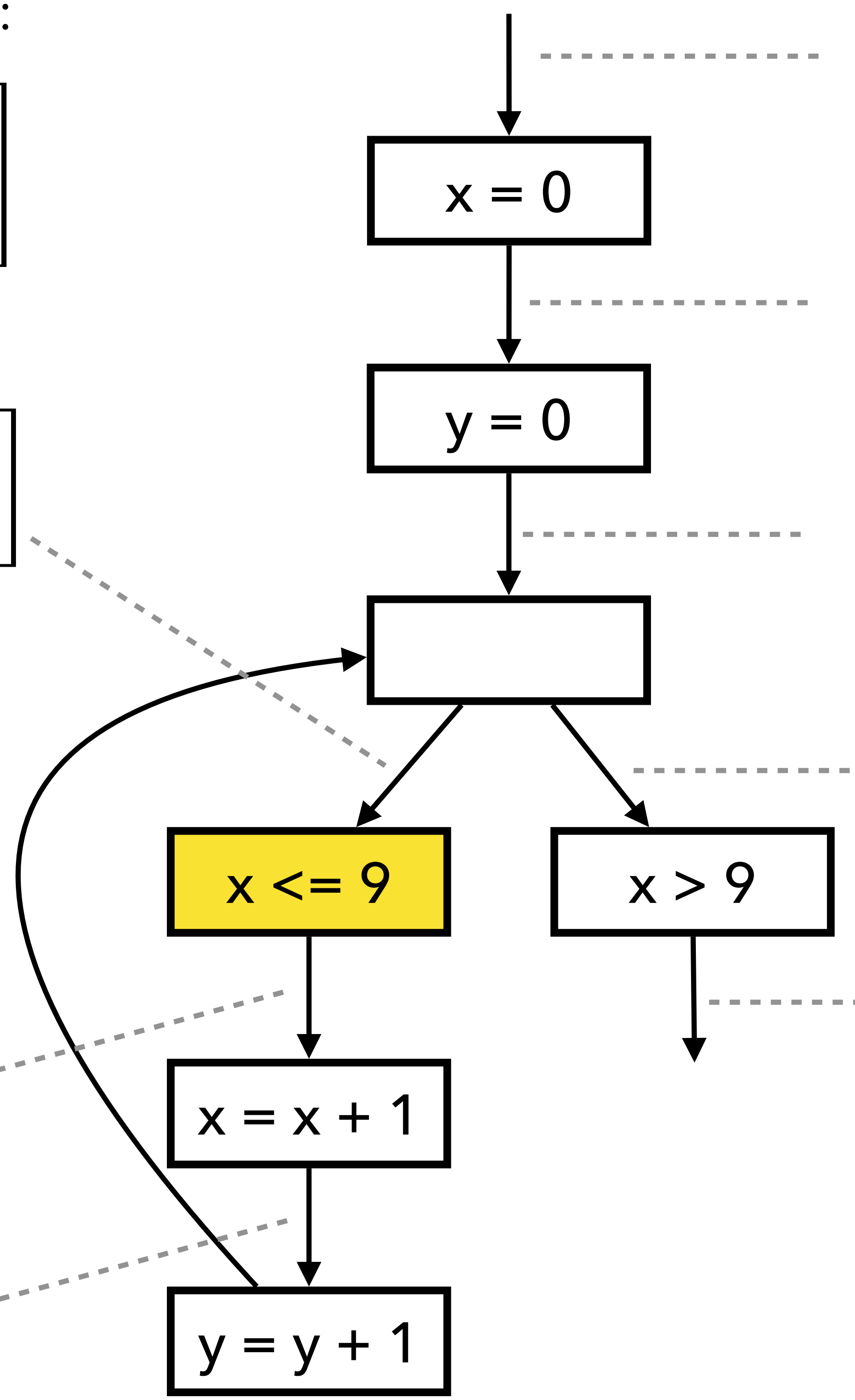
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \not\sqsubseteq \begin{bmatrix} 0 & \infty & \infty \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Fixed Point Comp. with Widening

1. Add constraint "x <= 9":

$$\begin{bmatrix}0 & \infty & \infty\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \to \begin{bmatrix}0 & 9 & \infty\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$



$$\begin{bmatrix}0 & \infty & \infty\\ \infty & 0 & \infty\\ \infty & \infty & 0\end{bmatrix}$$

$$\begin{bmatrix}0 & 0 & \infty\\ 0 & 0 & \infty\\ \infty & \infty & 0\end{bmatrix}$$

$$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}0 & \infty & \infty\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}0 & \infty & \infty\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$\perp$

$$\begin{bmatrix}0 & 1 & 1\\ -1 & 0 & 0\\ -1 & 0 & 0\end{bmatrix}$$

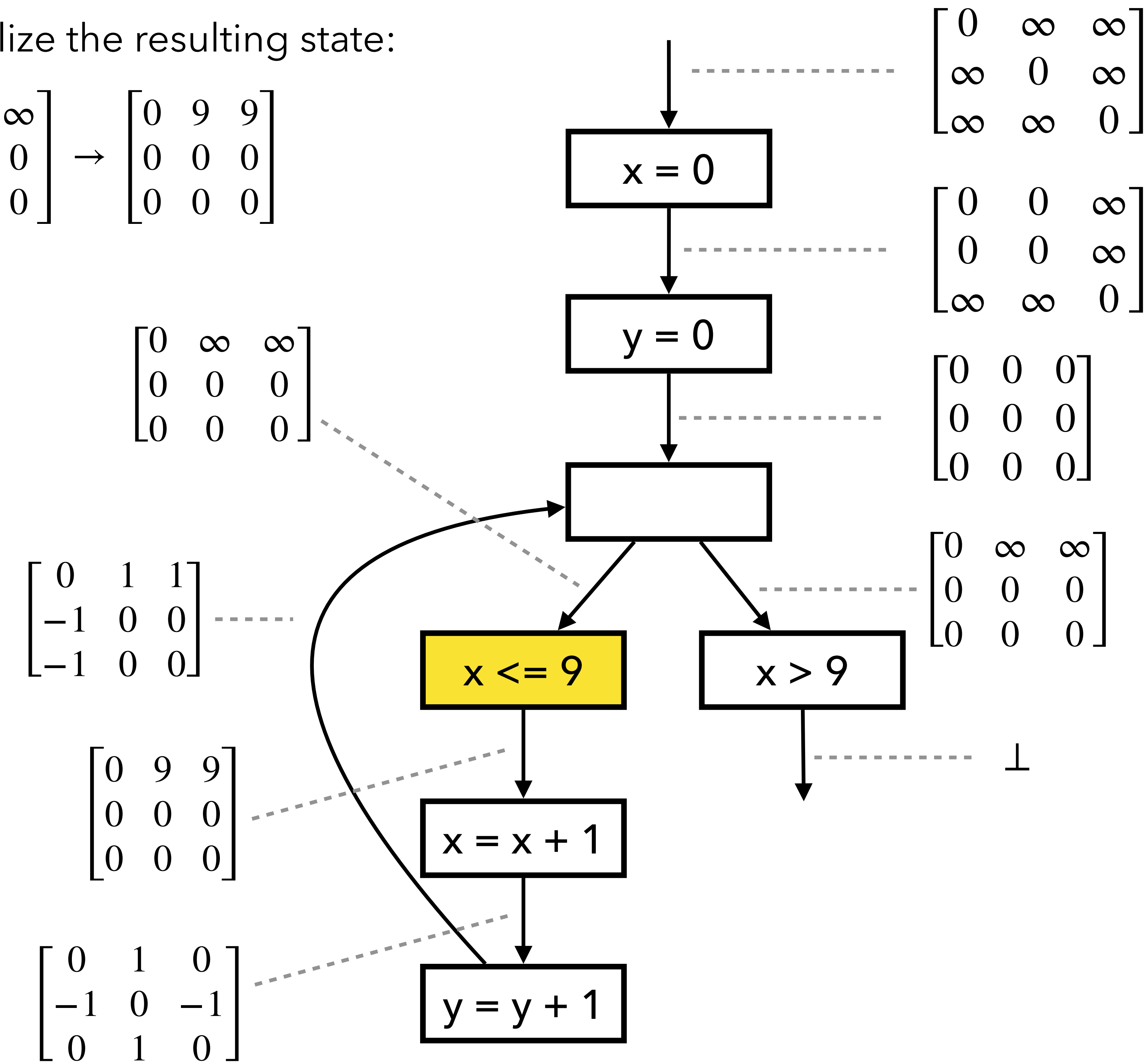
$$\begin{bmatrix}0 & 9 & 9\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & -1\\ 0 & 1 & 0\end{bmatrix}$$

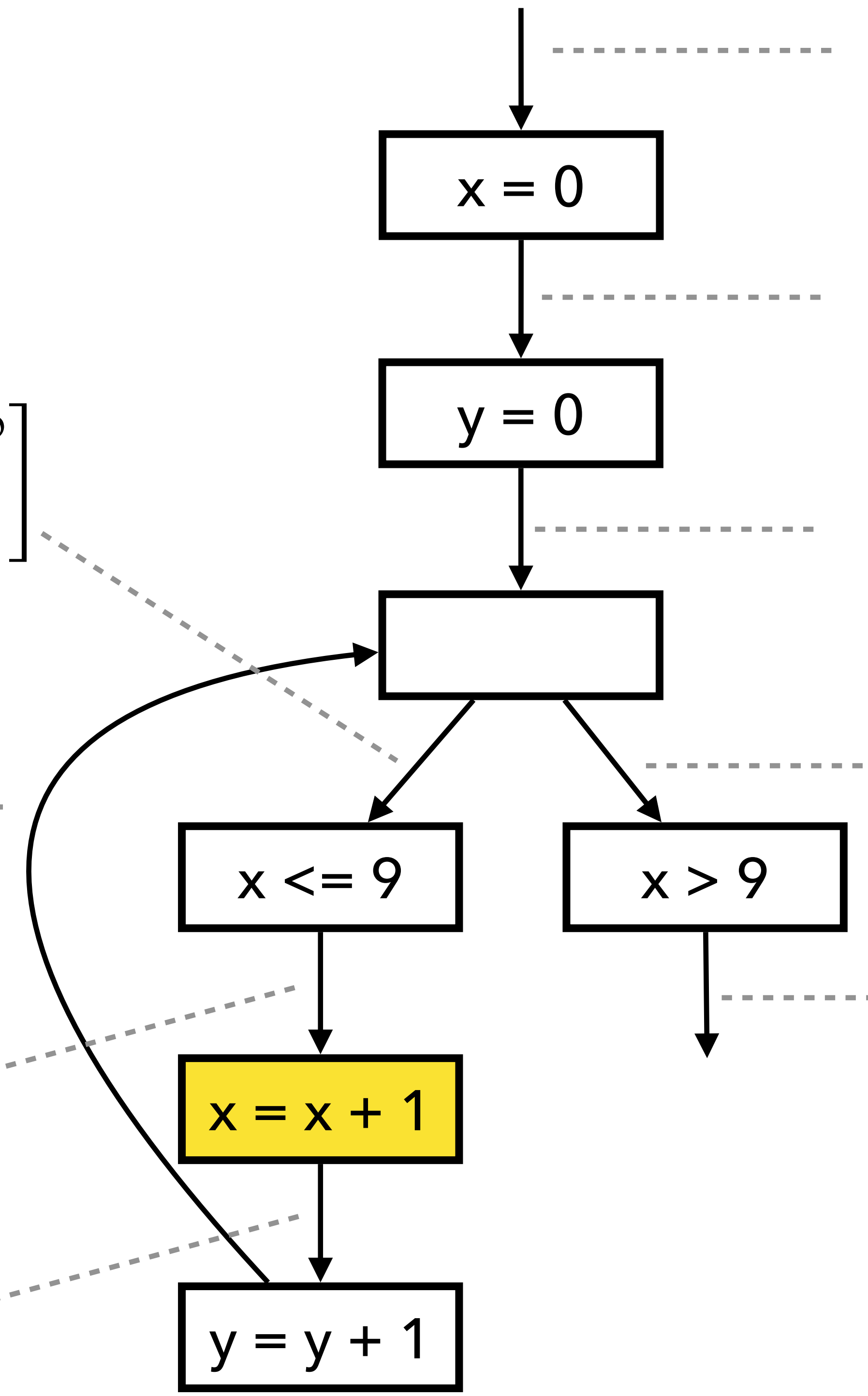
Fixed Point Comp. with Widening

2. Normalize the resulting state:

$$\begin{bmatrix} 0 & 9 & \infty \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 9 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Fixed Point Comp. with Widening



$$\begin{bmatrix} 0 & \infty & \infty \\ \infty & 0 & \infty \\ \infty & \infty & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & \infty \\ 0 & 0 & \infty \\ \infty & \infty & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \infty & \infty \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

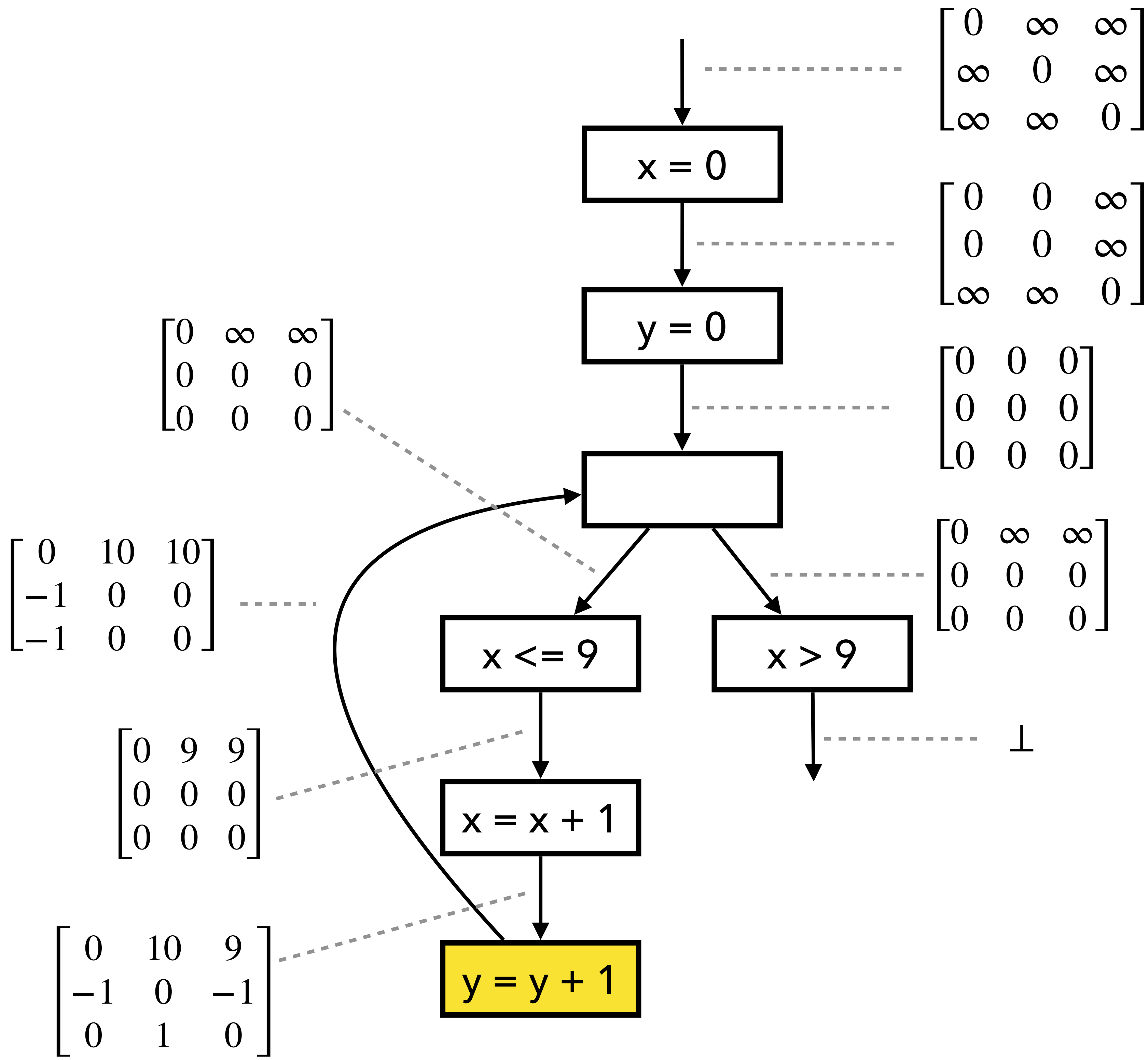
$$\begin{bmatrix} 0 & \infty & \infty \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 9 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 10 & 9 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

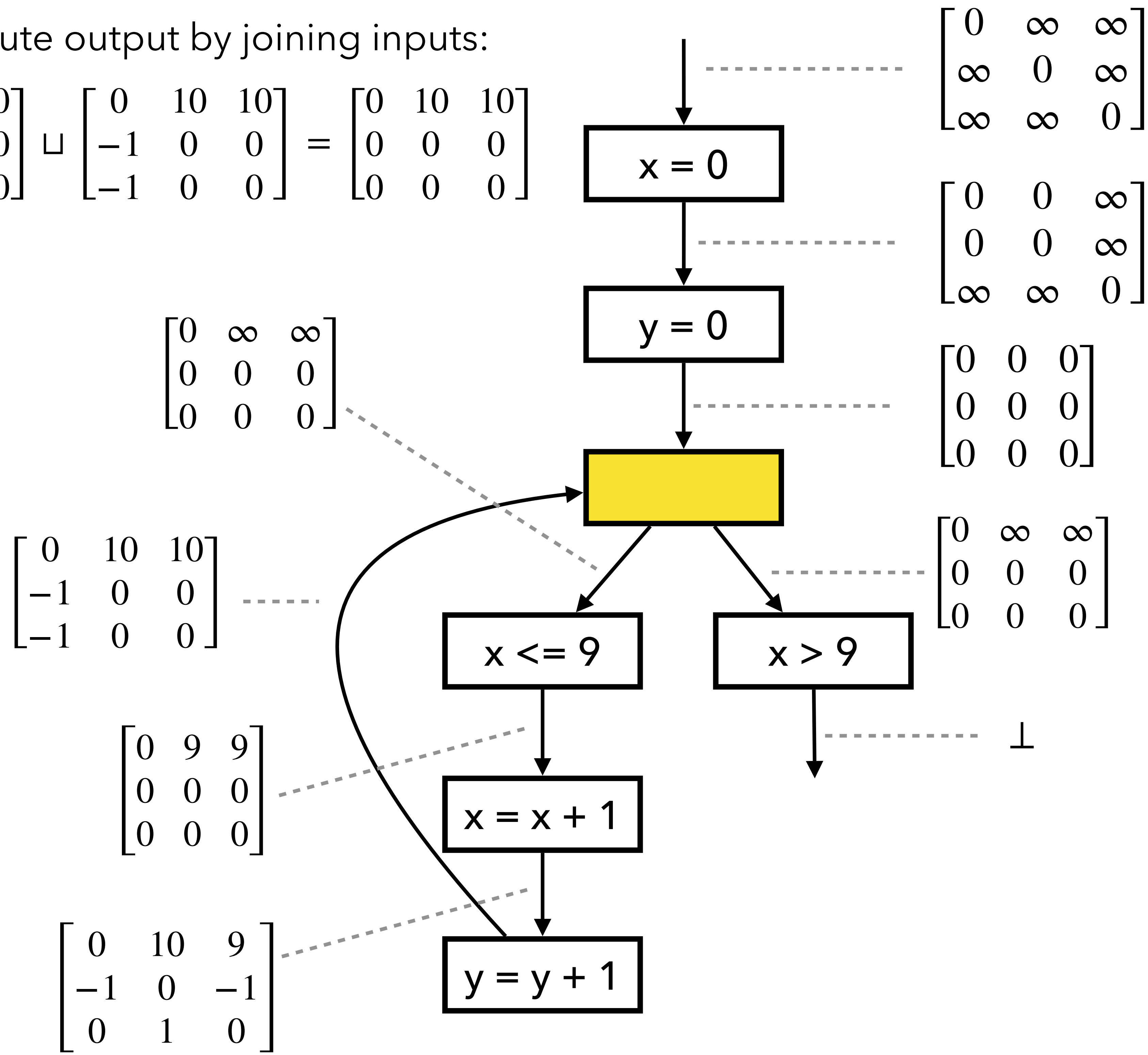
Fixed Point Comp. with Widening



Fixed Point Comp. with Widening

1. Compute output by joining inputs:

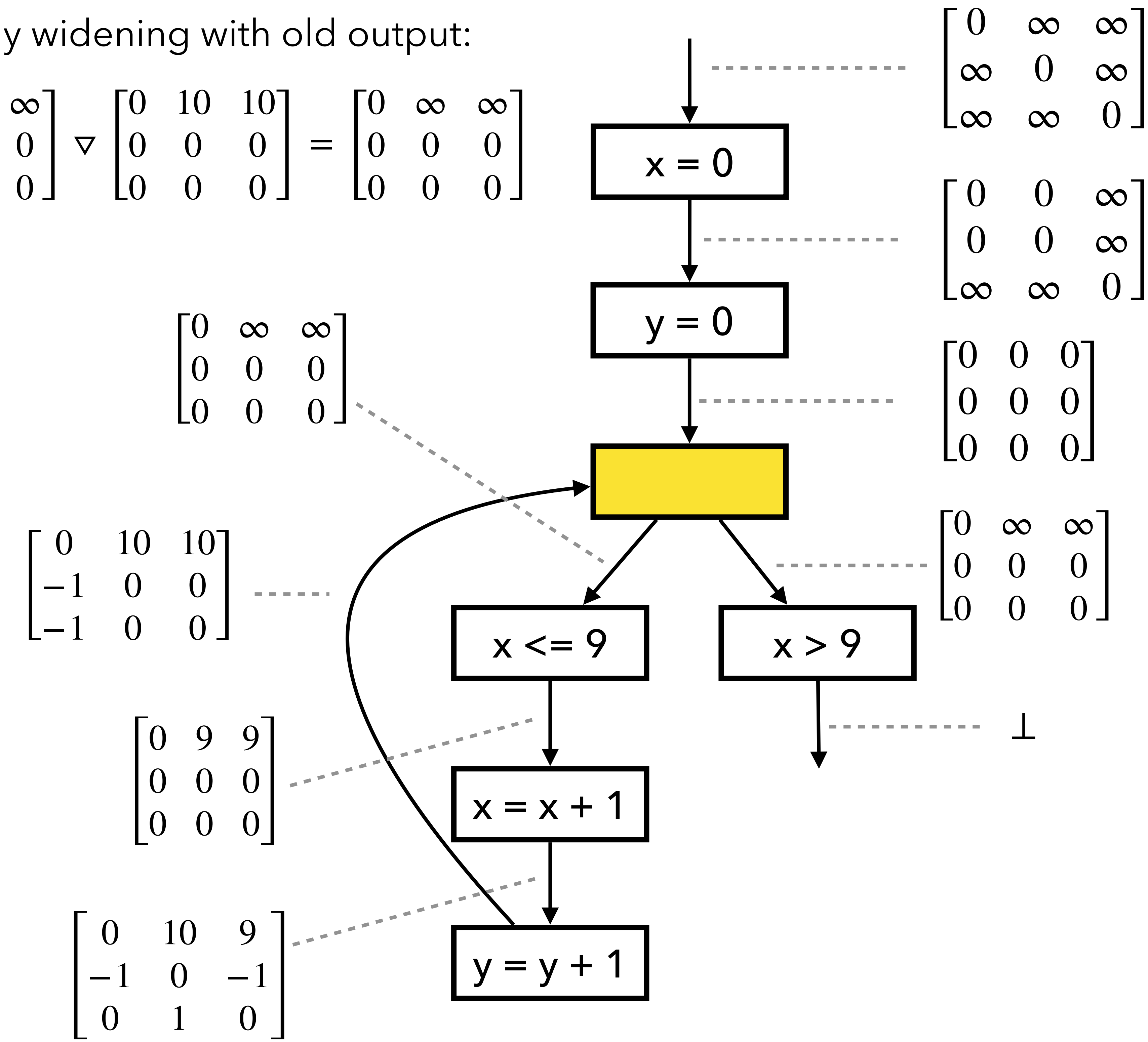
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sqcup \begin{bmatrix} 0 & 10 & 10 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 10 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Fixed Point Comp. with Widening

2. Apply widening with old output:

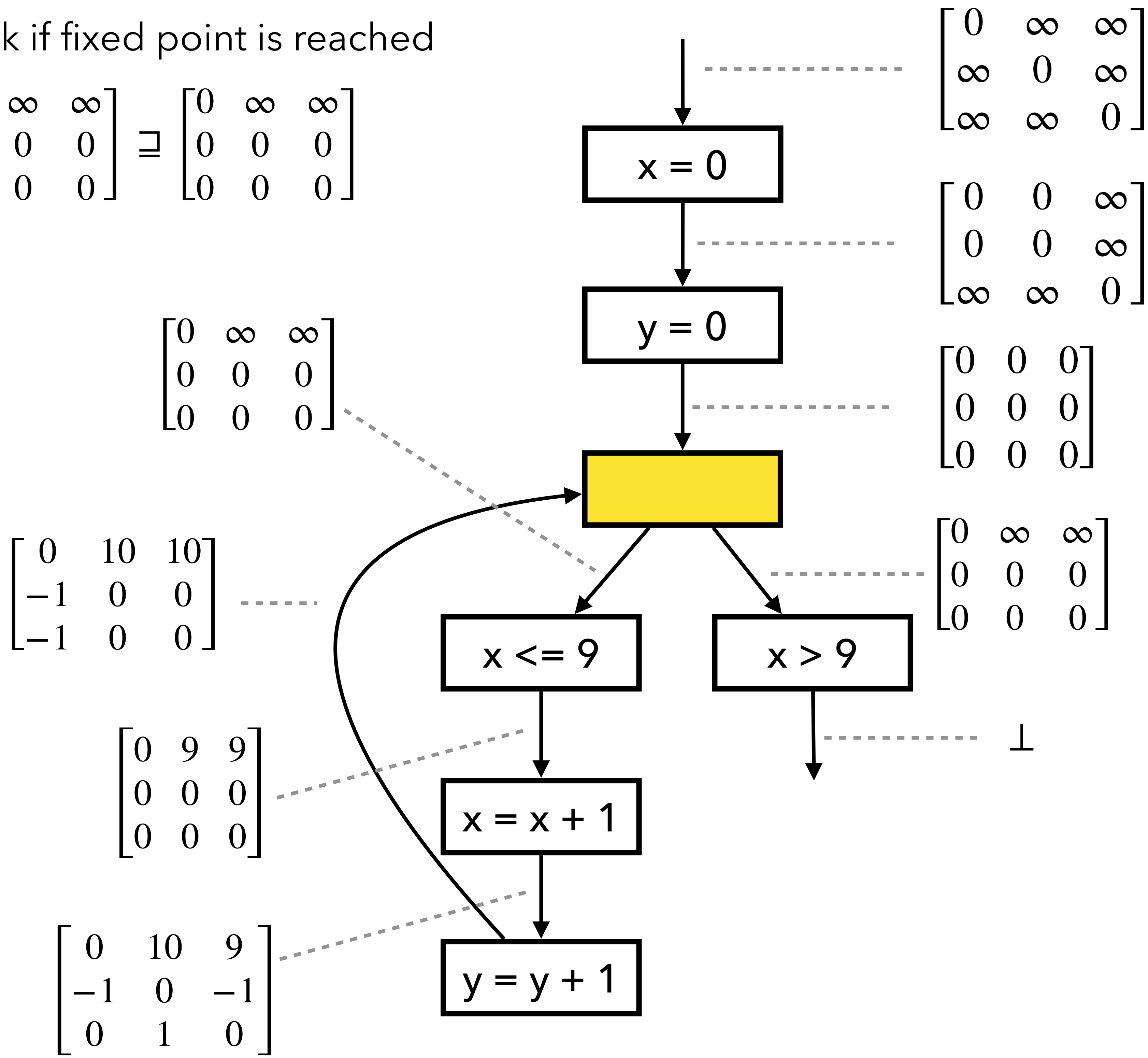
$$\begin{bmatrix} 0 & \infty & \infty \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \triangledown \begin{bmatrix} 0 & 10 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \infty & \infty \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Fixed Point Comp. with Widening

3. Check if fixed point is reached

$$\begin{bmatrix}0 & \infty & \infty \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix} \sqsupseteq \begin{bmatrix}0 & \infty & \infty \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}$$

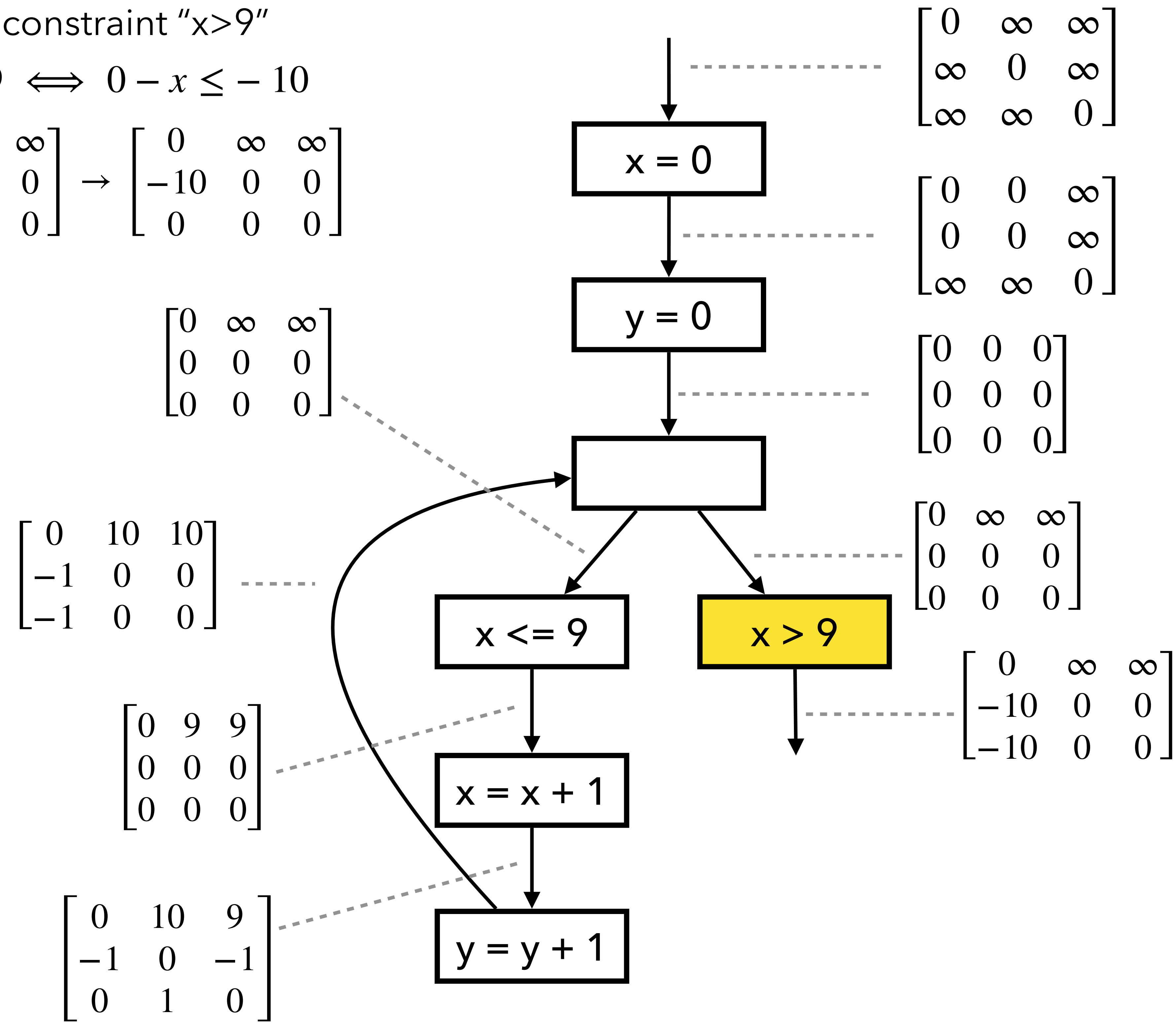


Fixed Point Comp. with Widening

1. Add constraint "x>9"

$$x > 9 \iff 0 - x \le -10$$

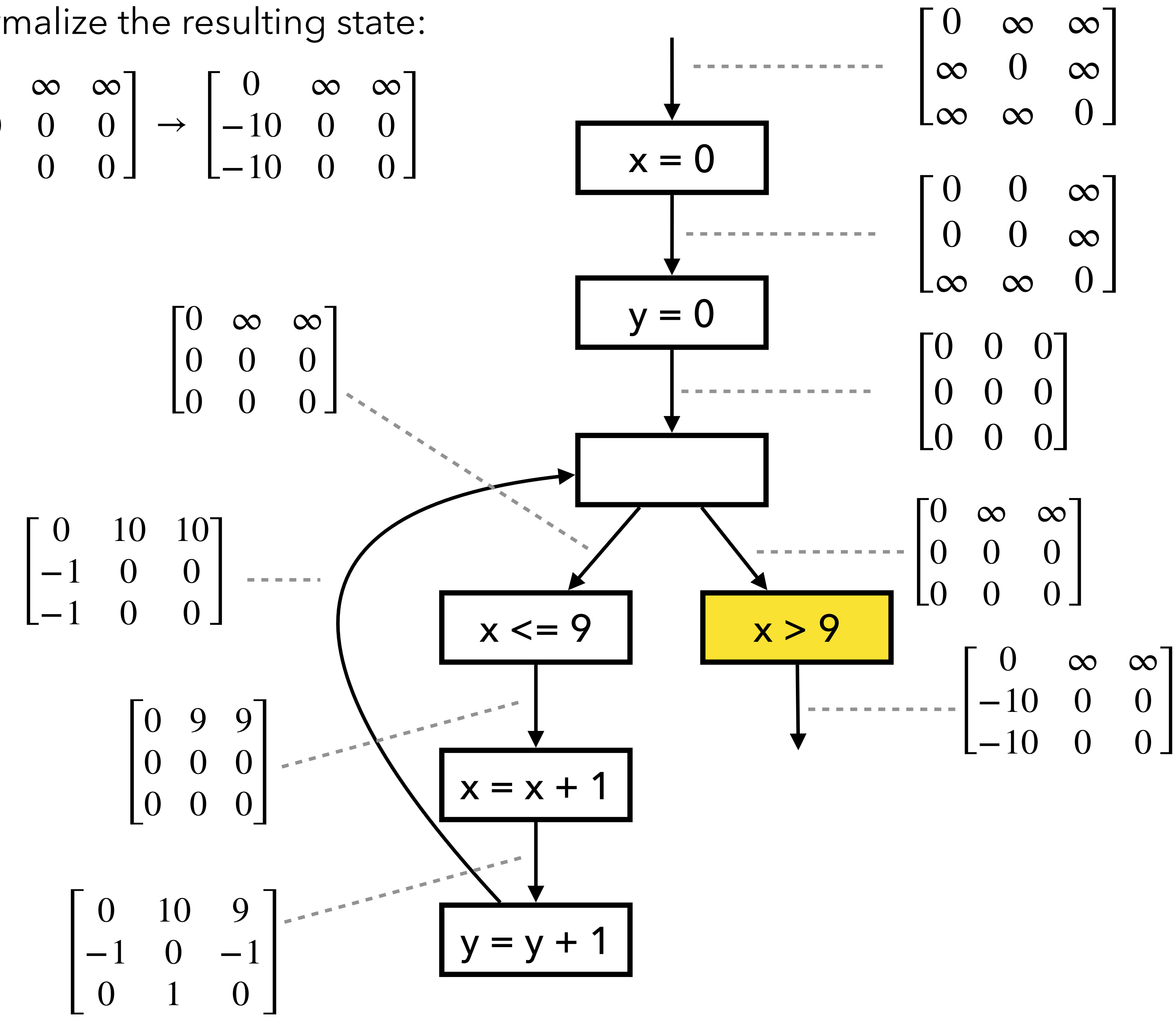
$$\begin{bmatrix} 0 & \infty & \infty \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & \infty & \infty \\ -10 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Fixed Point Comp. with Widening

2. Normalize the resulting state:

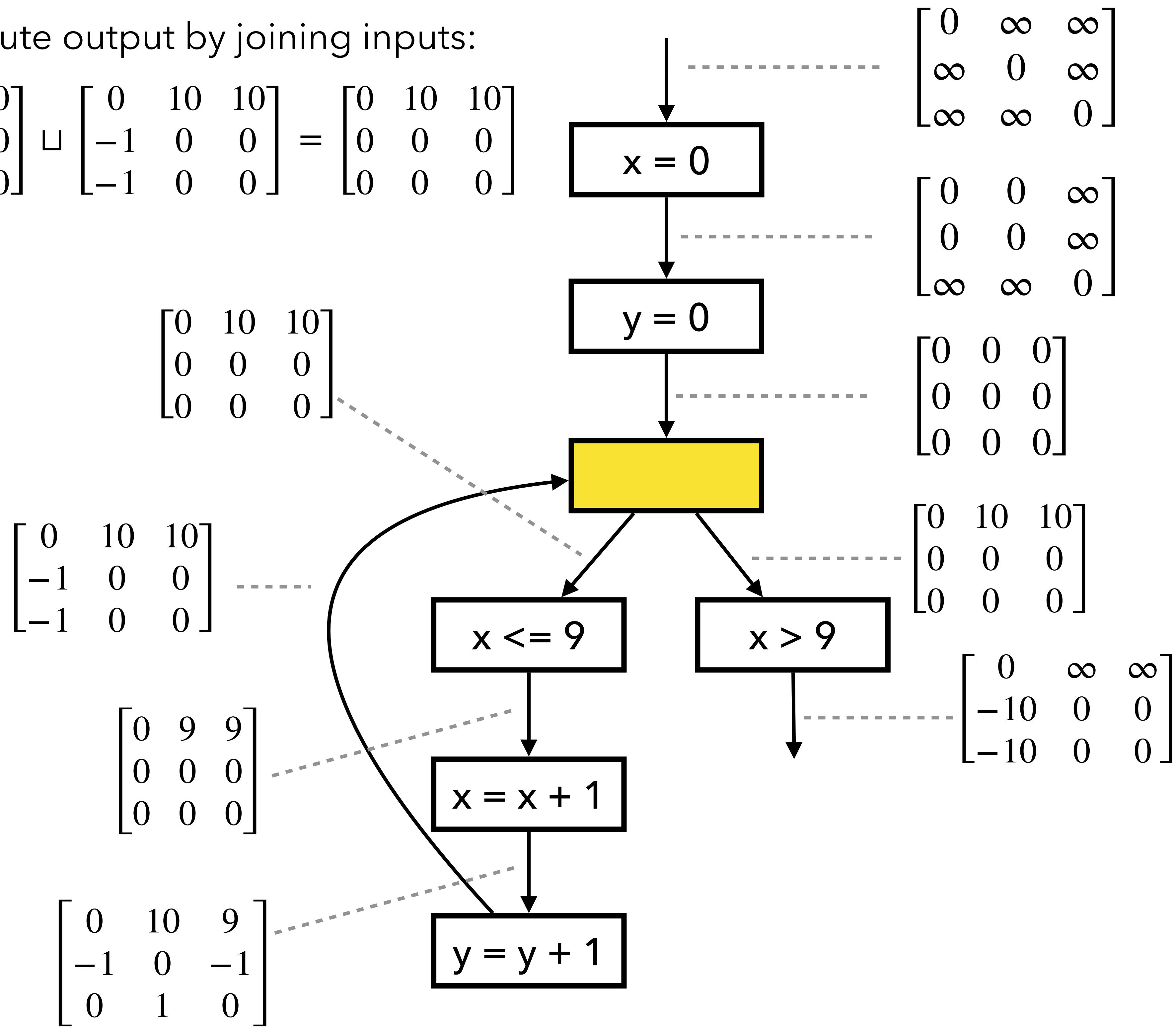
$$\begin{bmatrix} 0 & \infty & \infty \\ -10 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & \infty & \infty \\ -10 & 0 & 0 \\ -10 & 0 & 0 \end{bmatrix}$$



Fixed Point Comp. with Narrowing

1. Compute output by joining inputs:

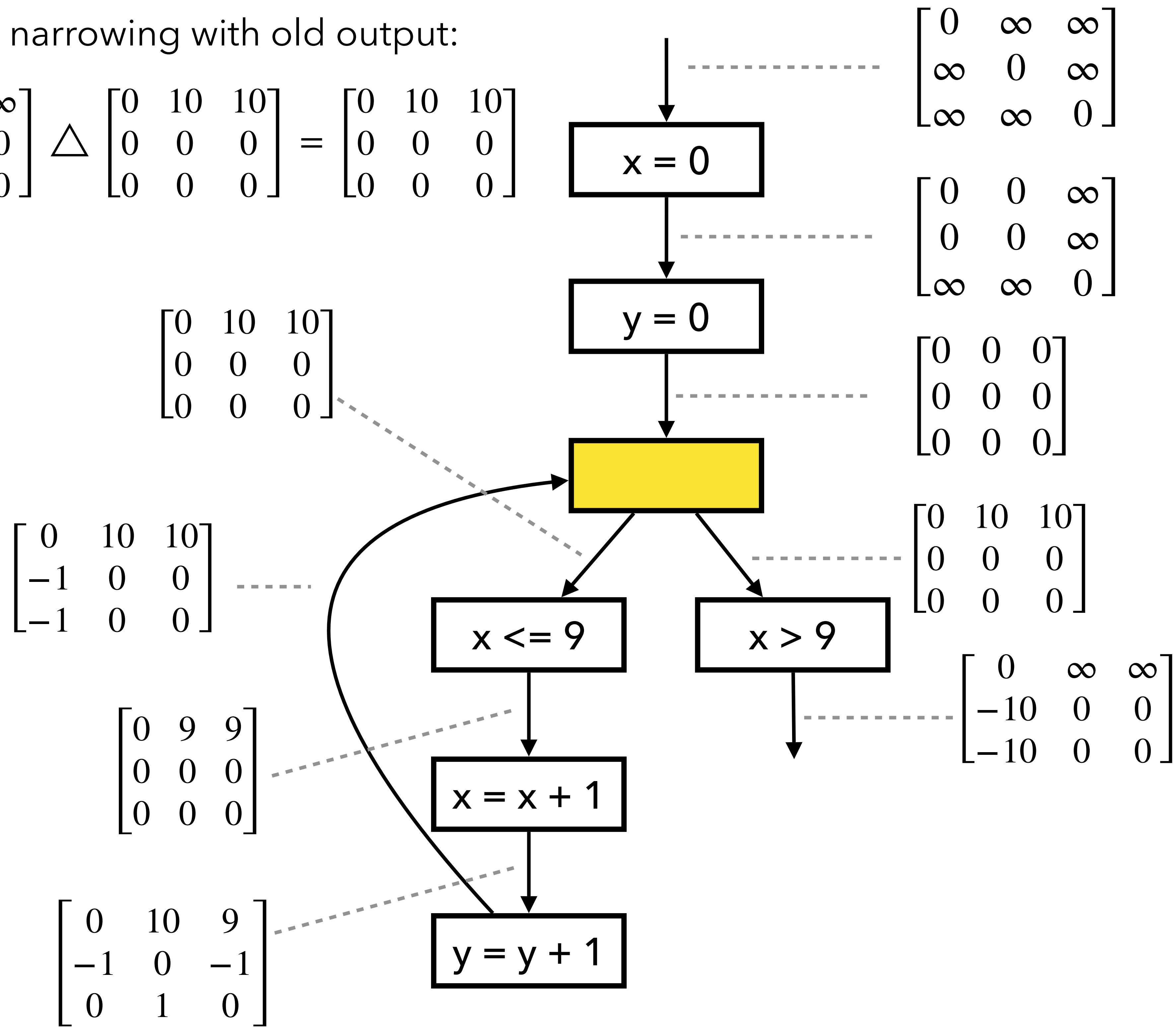
$$\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} \sqcup \begin{bmatrix}0 & 10 & 10\\-1 & 0 & 0\\-1 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 10 & 10\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$



Fixed Point Comp. with Narrowing

2. Apply narrowing with old output:

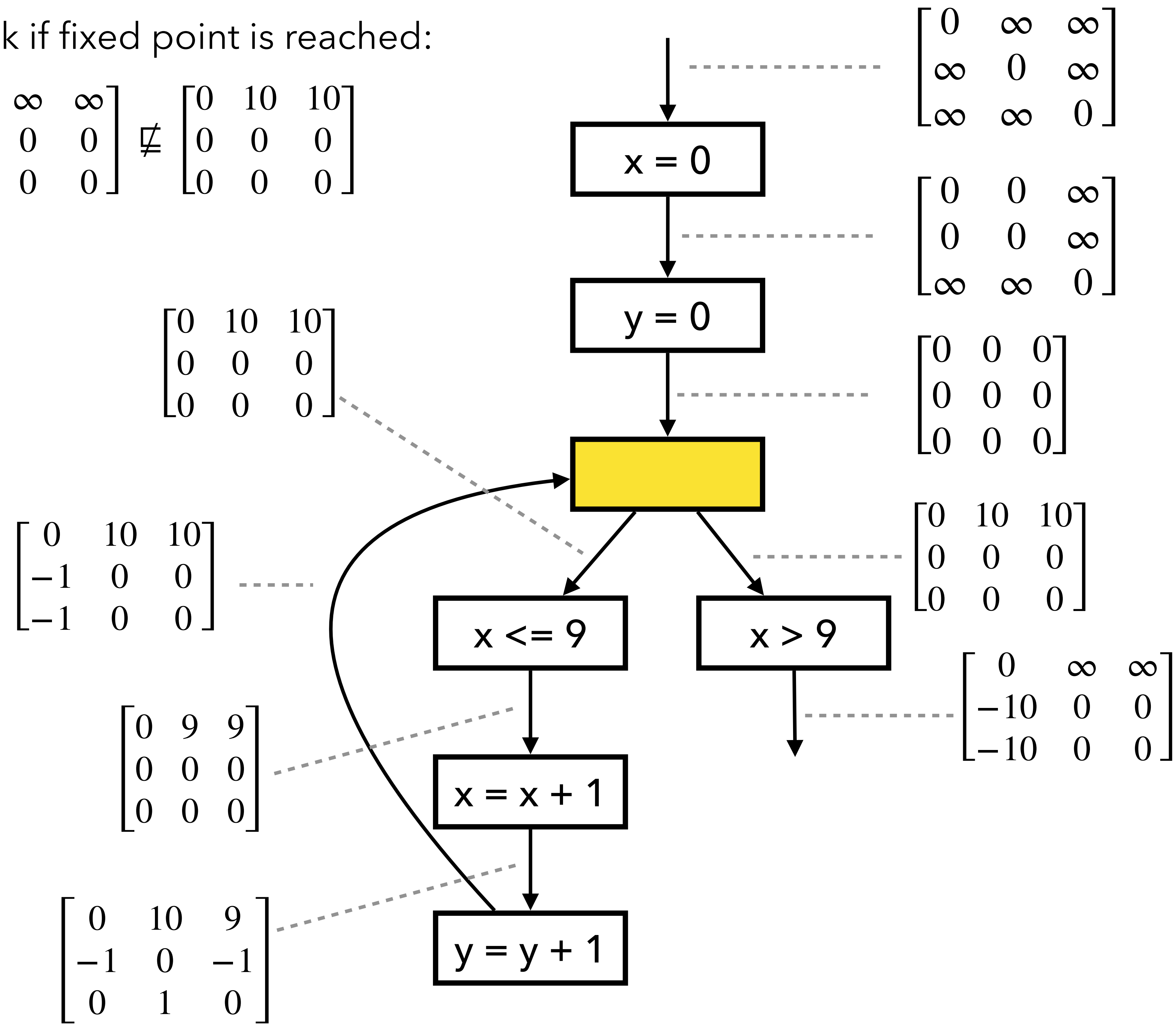
$$\begin{bmatrix} 0 & \infty & \infty \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \triangle \begin{bmatrix} 0 & 10 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 10 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



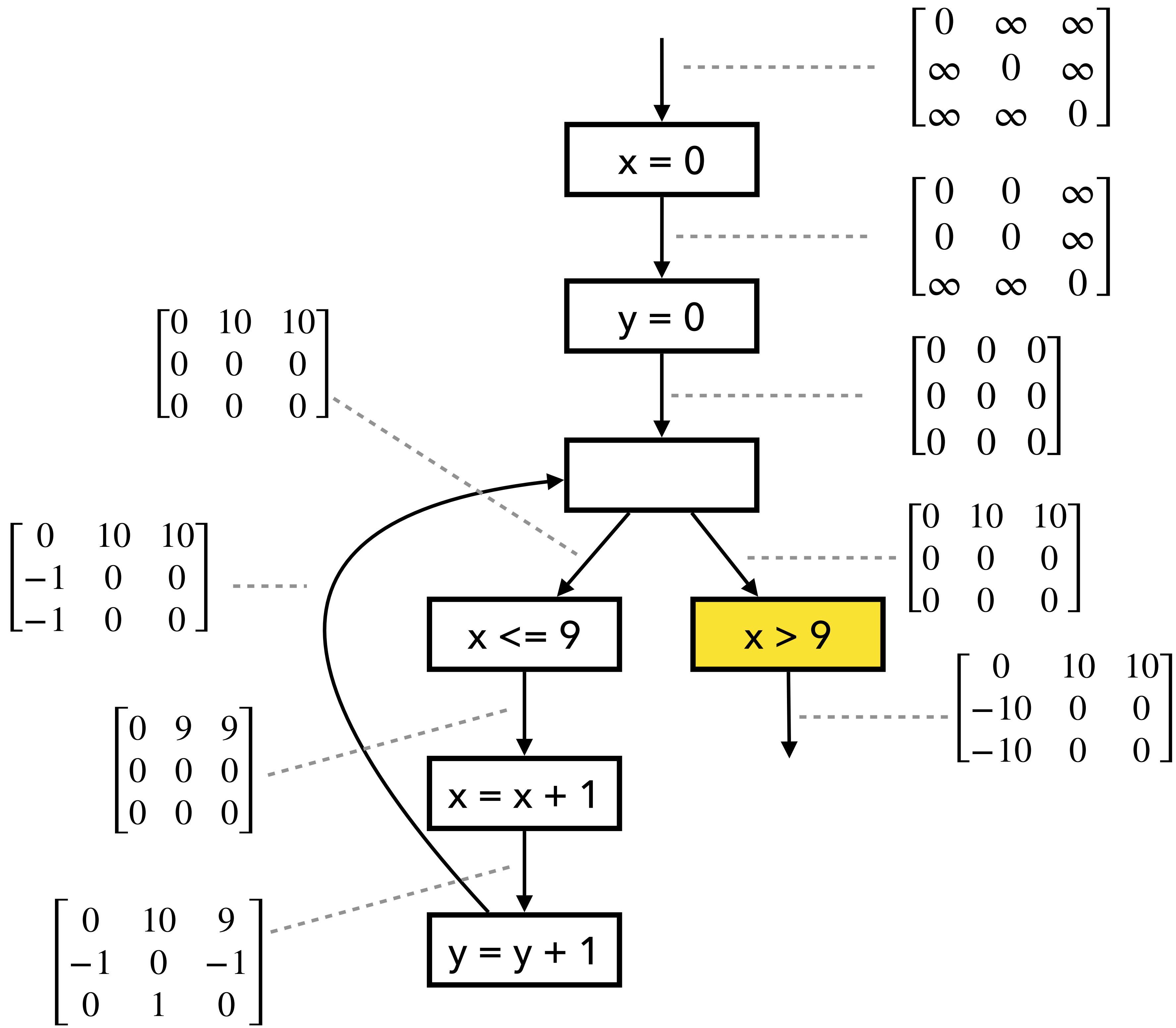
Fixed Point Comp. with Narrowing

3. Check if fixed point is reached:

$$\begin{bmatrix}0 & \infty & \infty\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \not\sqsubseteq \begin{bmatrix}0 & 10 & 10\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$



Fixed Point Comp. with Narrowing



Example

Describe how the zone analysis works for the following example.

```
// a >= 0, b >= 0
q = 0;
r = a;
while (r >= b) {
  r = r - b;
  q = q + 1;
}
assert(q >= 0);
assert(r >= 0);
```