# COSE212: Programming Languages

# Lecture 1 — Inductive Definitions (1)

Hakjoo Oh
2017 Fall

# Inductive Definitions

Inductive definition (induction) is widely used in the study of programming languages and computer science in general: e.g.,

- The syntax and semantics of programming languages
- Data structures (e.g. lists, trees, graphs, etc)

# Inductive Definitions

Inductive definition (induction) is widely used in the study of programming languages and computer science in general: e.g.,

- The syntax and semantics of programming languages
- Data structures (e.g. lists, trees, graphs, etc)

Induction is a technique for formally defining a set:

- The set is defined in terms of itself.
- The only way of defining an infinite set by a finite means.

# Inductive Definitions

Inductive definition (induction) is widely used in the study of programming languages and computer science in general: e.g.,

- The syntax and semantics of programming languages
- Data structures (e.g. lists, trees, graphs, etc)

Induction is a technique for formally defining a set:

- The set is defined in terms of itself.
- The only way of defining an infinite set by a finite means.

Three styles to inductive definition:

- Top-down
- Bottom-up
- Rules of inference

# Example (Top-Down)

## Definition ($S$)

A natural number $n$ is in $S$ if and only if

1. $n = 0$, or
2. $n - 3 \in S$.

# Example (Top-Down)

**Definition ($S$)**

A natural number $n$ is in $S$ if and only if

1. $n = 0$, or
2. $n - 3 \in S$.

- Inductive definition of a set of natural numbers: $S \subseteq \mathbb{N} = \{0, 1, \ldots\}$

# Example (Top-Down)

## Definition ($S$)

A natural number $n$ is in $S$ if and only if

1. $n = 0$, or
2. $n - 3 \in S$.

- Inductive definition of a set of natural numbers: $S \subseteq \mathbb{N} = \{0, 1, \ldots\}$
- $\{0, 3, 6, 9, \ldots\} \subseteq S$

# Example (Top-Down)

## Definition ($S$)

A natural number $n$ is in $S$ if and only if

1. $n = 0$, or
2. $n - 3 \in S$.

- Inductive definition of a set of natural numbers: $S \subseteq \mathbb{N} = \{0, 1, \ldots\}$
- $\{0, 3, 6, 9, \ldots\} \subseteq S$
- $\{0, 3, 6, 9, \ldots\} \supseteq S$

# Example (Top-Down)

## Definition ($S$)

A natural number $n$ is in $S$ if and only if

1. $n = 0$, or
2. $n - 3 \in S$.

- Inductive definition of a set of natural numbers: $S \subseteq \mathbb{N} = \{0, 1, \ldots\}$
- $\{0, 3, 6, 9, \ldots\} \subseteq S$
- $\{0, 3, 6, 9, \ldots\} \supseteq S$

$$S = \{0, 3, 6, 9, \ldots\}.$$

# Formal Proofs

**Lemma**

$\{0, 3, 6, 9, \ldots\} \subseteq S$

By induction. To show: $3k \in S$ for all $k \in \mathbb{N}$.

1. Base case: $3k \in S$ when $k = 0$.
2. Inductive case: Assume $3k \in S$ (Induction Hypothesis, I.H.). To show is $3 \cdot (k+1) \in S$, which holds because $3 \cdot (k+1) - 3 = 3k \in S$ by the induction hypothesis.

**Lemma**

$\{0, 3, 6, 9, \ldots\} \supseteq S$

By proof by contradiction. Let $n = 3k + q$ ($q = 1$ or $2$) and assume $n \in S$. By the definition of $S$, $n - 3$, $n - 6$, $\ldots, n - 3k \in S$. Thus, $S$ must include $1$ or $2$, a contradiction.

# A Bottom-up Definition

## Definition ($S$)

$S$ is the *smallest* set such that $S \subseteq \mathbb{N}$ and $S$ satisfies the following two conditions:

1. $0 \in S$, and
2. if $n \in S$, then $n + 3 \in S$.

# A Bottom-up Definition

**Definition ($S$)**

$S$ is the *smallest* set such that $S \subseteq \mathbb{N}$ and $S$ satisfies the following two conditions:

1. $0 \in S$, and
2. if $n \in S$, then $n + 3 \in S$.

- The two conditions imply $\{0, 3, 6, 9, \ldots\} \subseteq S$.

# A Bottom-up Definition

**Definition ($S$)**

$S$ is the *smallest* set such that $S \subseteq \mathbb{N}$ and $S$ satisfies the following two conditions:

1. $0 \in S$, and
2. if $n \in S$, then $n + 3 \in S$.

- The two conditions imply $\{0, 3, 6, 9, \ldots\} \subseteq S$.
- The two conditions do not imply $\{0, 3, 6, 9, \ldots\} \supseteq S$, e.g., $\mathbb{N}$.

# A Bottom-up Definition

**Definition ($S$)**

$S$ is the *smallest* set such that $S \subseteq \mathbb{N}$ and $S$ satisfies the following two conditions:

1. $0 \in S$, and
2. if $n \in S$, then $n + 3 \in S$.

- The two conditions imply $\{0, 3, 6, 9, \ldots\} \subseteq S$.
- The two conditions do not imply $\{0, 3, 6, 9, \ldots\} \supseteq S$, e.g., $\mathbb{N}$.
- By requiring $S$ to be the **smallest** such a set,

$$S = \{0, 3, 6, 9, \ldots\}.$$

# A Bottom-up Definition

**Definition ($S$)**

$S$ is the *smallest* set such that $S \subseteq \mathbb{N}$ and $S$ satisfies the following two conditions:

1. $0 \in S$, and
2. if $n \in S$, then $n + 3 \in S$.

- The two conditions imply $\{0, 3, 6, 9, \ldots\} \subseteq S$.
- The two conditions do not imply $\{0, 3, 6, 9, \ldots\} \supseteq S$, e.g., $\mathbb{N}$.
- By requiring $S$ to be the **smallest** such a set,

$$S = \{0, 3, 6, 9, \ldots\}.$$

- The smallest set satisfying the conditions is unique.
  - Proof) If $S_1$ and $S_2$ satisfy the conditions and are both smallest, then $S_1 \subseteq S_2$ and $S_2 \subseteq S_1$. Therefore, $S_1 = S_2$ ($\subseteq$ is anti-symmetric).

# Rules of Inference

$$\frac{\boldsymbol{A}}{\boldsymbol{B}}$$

- $\boldsymbol{A}$: hypothesis (antecedent)
- $\boldsymbol{B}$: conclusion (consequent)
- "if $\boldsymbol{A}$ is true then $\boldsymbol{B}$ is also true".
- $\overline{\boldsymbol{B}}$: axiom.

# Defining a Set by Rules of Inferences

**Definition**

$$\frac{}{0 \in S}$$

$$\frac{n \in S}{(n+3) \in S}$$

Interpret the rules as follows:

"A natural number $n$ is in $S$ iff $n \in S$ can be derived from the axiom by applying the inference rules finitely many times"

# Defining a Set by Rules of Inferences

## Definition

$$\frac{}{\mathbf{0} \in \boldsymbol{S}}$$

$$\frac{\boldsymbol{n} \in \boldsymbol{S}}{(\boldsymbol{n} + \mathbf{3}) \in \boldsymbol{S}}$$

Interpret the rules as follows:

"A natural number $\boldsymbol{n}$ is in $\boldsymbol{S}$ iff $\boldsymbol{n} \in \boldsymbol{S}$ can be derived from the axiom by applying the inference rules finitely many times"

ex) $\mathbf{3} \in \boldsymbol{S}$ because

$$\cfrac{\cfrac{}{\mathbf{0} \in \boldsymbol{S}}\ \text{the axiom}}{\mathbf{3} \in \boldsymbol{S}}\ \text{the second rule}$$

# Defining a Set by Rules of Inferences

**Definition**

$$\frac{}{0 \in S}$$

$$\frac{n \in S}{(n+3) \in S}$$

Interpret the rules as follows:

"A natural number $n$ is in $S$ iff $n \in S$ can be derived from the axiom by applying the inference rules finitely many times"

ex) $3 \in S$ because

$$\cfrac{\overline{0 \in S}\ \text{the axiom}}{3 \in S}\ \text{the second rule}$$

Note that this interpretation enforces that $S$ is the smallest set closed under the inference rules.

# Exercises

1. What set is defined by the following inductive rules?

$$\frac{}{\boldsymbol{3}} \qquad \frac{\boldsymbol{x} \quad \boldsymbol{y}}{\boldsymbol{x+y}}$$

# Exercises

1. What set is defined by the following inductive rules?

$$\frac{}{\mathbf{3}} \qquad \frac{\boldsymbol{x} \quad \boldsymbol{y}}{\boldsymbol{x} + \boldsymbol{y}}$$

2. What set is defined by the following inductive rules?

$$\frac{}{()} \qquad \frac{\boldsymbol{s}}{(\boldsymbol{s})} \qquad \frac{\boldsymbol{s}}{\boldsymbol{s}\ \boldsymbol{s}}$$

# Exercises

1. What set is defined by the following inductive rules?

$$\frac{}{3} \qquad \frac{x \quad y}{x + y}$$

2. What set is defined by the following inductive rules?

$$\frac{}{()} \qquad \frac{s}{(s)} \qquad \frac{s}{s\ s}$$

3. Define the following set as rules of inference:

$$S = \{a, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, bab, bba, bbb, \ldots\}$$

# Exercises

1. What set is defined by the following inductive rules?

$$\frac{}{3} \qquad \frac{x \quad y}{x+y}$$

2. What set is defined by the following inductive rules?

$$\frac{}{()} \qquad \frac{s}{(s)} \qquad \frac{s}{s\ s}$$

3. Define the following set as rules of inference:

$$S = \{a, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, bab, bba, bbb, \ldots\}$$

4. Define the following set as rules of inference:

$$S = \{x^n y^{n+1} \mid n \in \mathbb{N}\}$$

# Summary

In inductive definitions, a set is defined in terms of itself. Three styles:

- Top-down
- Bottom-up
- Rules of inference

In PL, we mainly use the rules-of-inference method.