Homework 1
AAA 616: Program Analysis, Fall 2016

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Due: 09/27 (in class)

Problem 1 Consider the poset of 'vertical natural numbers', denoted $(\Omega, \sqsubseteq)$, where

$$\Omega = \mathbb{N} \cup \{\omega\}, \qquad d \sqsubseteq d' \text{ iff } \begin{cases} & d, d' \in \mathbb{N} \ \wedge \ d \leq d' \\ \text{or} & d \in \mathbb{N} \ \wedge \ d' = \omega \\ \text{or} & d = d' = \omega \end{cases}$$

Prove that $(\Omega, \sqsubseteq)$ is a CPO.

Problem 2 Let $(X \hookrightarrow Y, \sqsubseteq)$ be the poset of all partial functions from a set X to a set Y, equipped with the partial order

$$\mathsf{dom}(f) \subseteq \mathsf{dom}(g) \ \wedge \ \forall x \in \mathsf{dom}(f).\ f(x) = g(x).$$

The least upper bound of a chain $Y \subseteq (X \hookrightarrow Y)$, i.e., $\bigsqcup Y$, is given by the partial function f with $\mathsf{dom}(f) = \bigcup_{f_i \in Y} \mathsf{dom}(f_i)$ and

$$f(x) = \begin{cases} f_n(x) & \cdots x \in \mathsf{dom}(f_i) \text{ for some } f_i \in Y \\ \mathsf{undef} & \cdots \textit{otherwise} \end{cases}$$

Prove that $(X \hookrightarrow Y, \sqsubseteq)$ is a CPO but not a complete lattice.

Problem 3 Consider the program:

$$\texttt{while}\ (x > 0)\ (y := x * y; x := x - 1)$$

- Define the function F for the program.
- Prove that

  $$g = \lambda s. \begin{cases} s & \cdots s(x) \leq 0 \\ s[x \mapsto 0, y \mapsto s(x)!s(y)] & \cdots s(x) > 0 \end{cases}$$

  is a fixed point of F.