COSE215: Theory of Computation

Lecture 14 — Properties of Context-Free Languages (2)

Hakjoo Oh
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The Pumping Lemma for RLs

Theorem (Pumping Lemma for RLs)

For any regular language L there exists an integer n, such that for all $x \in L$ with $|x| \geq n$, there exist $u, v, w \in \Sigma^*$, such that

1. $x = uvw$
2. $|uv| \leq n$
3. $|v| \geq 1$
4. *for all* $i \geq 0$, $uv^iw \in L$.

"For any RL, we can find one small string to pump"

The Pumping Lemma for CFLs

Theorem (Pumping Lemma for CFLs)

For any context-free language L there exists an integer n, such that for all $z \in L$ with $|z| \geq n$, there exist $u, v, w, x, y \in \Sigma^*$, such that

1. $z = uvwxy$
2. $|vwx| \leq n$
3. $|vx| \geq 1$
4. *for all* $i \geq 0$, $uv^iwx^iy \in L$.

"For any CFL, we can find two small strings to pump in tandem"

Example

A context-free language:

$$L = \{0^k 1^k \mid k \geq 0\}$$

Thus the pumping lemma holds with $n = 2$:

- Any $z \in L$ with $|z| \geq 2$ satisfies the pumping lemma.
- E.g., $z = 01$:
  - $u = \epsilon, v = 0, w = \epsilon, x = 1, y = \epsilon$.
  - $z = uvwxy$
  - $|vwx| \leq 2$
  - $|vx| \geq 1$
  - for all $i \geq 0$, $uv^i wx^i y \in L$
- E.g., $z = 0011$:
  - $u = 0, v = 0, w = \epsilon, x = 1, y = 1$.
  - $z = uvwxy$
  - $|vwx| \leq 2$
  - $|vx| \geq 1$
  - for all $i \geq 0$, $uv^i wx^i y \in L$

Proving languages not to be context-free

- If L is context-free, L satisfies the pumping lemma.
- Although L satisfies the pumping lemma, L may not be context-free.
- If L does not satisfy pumping lemma, then L is not context-free.

P.L. can be used only for proving languages not to be context-free.

Example 1

Prove that $L = \{0^n 1^n 2^n \mid n \geq 1\}$ is not context-free.

- Show that pumping lemma (P.L.) does not hold.
- If L is context-free, then by P.L. there exists n such that ...
- Now let $z = 0^n 1^n 2^n$
- $z \in L$ and $|z| \geq n$, so by P.L. there exist u, v, w, x, y such that (1)–(4) hold.
- We show that for all u, v, w, x, y (1)–(4) do not all hold.
- If (1), (2), (3) hold then $z = 0^n 1^n 2^n = uvwxy$ with $|vwx| \leq n$ and $|vx| \geq 1$.
- So, vwx cannot involve both 0's and 2's, since $|vwx| \leq n$.
  1. vwx has no 2's (y has n 2's). Then (4) fails for $i = 0$: $uv^0wx^0y = uwy$ has n 2's but fewer 0's or 1's, since $|vx| \geq 1$. Contradiction.
  2. vwx has no 0's (u has n 0's). Then (4) fails for $i = 0$. Contradiction.

Example 2

Prove that $L = \{0^i 1^j 2^i 3^j \mid i \geq 1, j \geq 1\}$ is not context-free.

- Show that pumping lemma (P.L.) does not hold.
- If L is context-free, then by P.L. there exists n such that ...
- Now let $z = 0^n 1^n 2^n 3^n$
- $z \in L$ and $|z| \geq n$, so by P.L. there exist u, v, w, x, y such that (1)–(4) hold.
- We show that for all u, v, w, x, y (1)–(4) do not all hold.
- If (1), (2), (3) hold then $z = 0^n 1^n 2^n 3^n = uvwxy$ with $|vwx| \leq n$ and $|vx| \geq 1$.
- So, vwx has either only one symbol or straddles two adjacent symbols, since $|vwx| \leq n$.
  1. vwx has only one symbol. Then (4) fails for $i = 0$: uwy has n of three different symbols and fewer than n of the fourth symbol. Thus, $uwy \notin L$. Contradiction.
  2. vwx straddles two symbols, say 1's and 2's. Then, uwy is missing either some 1's or some 2's. Thus, $uwy \notin L$. Contradiction.

Exercise

Prove that $L = \{ww \mid w \in \{0,1\}^*\}$ is not context-free.

Closure Properties of CFLs

Regular languages are closed under:

- union,
- intersection,
- concatenation,
- closure,
- complementation, ...

Context-free languages are closed under:

- union,
- concatenation,
- closure

But, CFLs are not closed under intersection and complementation.

CFLs are not closed under intersection

The languages

$$L_1 = \{a^n b^n c^m \mid n \geq 0, m \geq 0\}$$

$$L_2 = \{a^n b^m c^m \mid n \geq 0, m \geq 0\}$$

are context-free, but their intersection

$$L_1 \cap L_2 = \{a^n b^n c^n \mid n \geq 0\}$$

is not context-free.

CFLs are not closed under complementation

Suppose that CFLs are closed under complementation. Then, a contradiction is derived from

$$L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$$

and the fact that CFLs are closed under union.

cf) CFLs are closed under regular intersection

Theorem

If L_1 is a context-free language and L_2 is a regular language, then $L_1 \cap L_2$ is context-free.

Examples:

- $L_1 = \{a^n b^n \mid n \geq 0\}$ is context-free and $L_2 = \{a^{100} b^{100}\}$ is regular. Thus, $L = \{a^n b^n \mid n \geq 0, n \neq 100\}$ is context-free.
- $L = \{w \in \{a, b, c\}^* \mid n_a(w) = n_b(w) = n_c(w)\}$ is not context-free, because

$$L \cap L(a^* b^* c^*) = \{a^n b^n c^n \mid n \geq 0\}$$

is not context-free.